Finished Friday on localizations. Focusing on $\mathcal{O}_\wp = \mathcal{O}\cdot(\mathcal{O}\setminus\wp)^{-1}$

$$= \left\{ \frac{a}{s} \;\middle|\; a\in\mathcal{O},\ s\in\mathcal{O}\setminus\wp \right\}$$

e.g. $\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \;\middle|\; \begin{array}{c} a,b\in\mathbb{Z} \\ p\nmid b \end{array} \right\}$

Proposition: If $\mathcal{O}$ Noetherian domain, then $\mathcal{O}$ Dedekind $\iff$ $\mathcal{O}_\wp$ is a discrete valuation ring $\forall\ \wp\neq 0$ prime ideals.

pf: ($\Rightarrow$) Show if $\mathcal{O}$ Dedekind, then $\mathcal{O}_\wp$ Dedekind (use this correspondence on ideals, for Noetherian – use "every submodule is fin. generated" criterion)

Worked for any mult. closed subset

$\mathcal{O}_\wp$ Dedekind $\Rightarrow$ all $(\neq 0)$ prime ideals maximal.

Know $\mathcal{O}_\wp$ local $\Rightarrow$ unique maximal ideal

Together they imply unique non-zero prime ideal.

$\Rightarrow$ $\mathcal{O}_\wp$ is P.I.D.

($\Leftarrow$) Follows from fact that $\mathcal{O} = \bigcap_{\wp\neq 0} \mathcal{O}_\wp$. ($\subseteq$ clear. Need to show $\supseteq$) Not hard.

As consequence, given Dedekind domain $\mathcal{O}_K =: \mathcal{O}$, then ideal factorization in $\mathcal{O}_\wp$ is wonderfully boring. Ideals are of form, $x\in K^\times$,

$$(x) = \wp_1^{e_1}\cdots\wp_r^{e_r} \quad \text{in } \mathcal{O}_K.$$

If we localize at one of $\wp_i$: $\mathcal{O}_{\wp_i} = \wp_j\mathcal{O}_{\wp_i}$ $i\neq j$, so

$$x\cdot\mathcal{O}_{\wp_i} = \wp_i^{e_i}\mathcal{O}_{\wp_i}$$ and valuation of x in $\mathcal{O}_{\wp_i}$ is e_i.

Finally, consider localization at $\bigcup_{x \in X} \mathfrak{p}_x$. Following Neukirch,

write $\mathcal{O}(X) = \{ \frac{f}{g} \mid f, g \in \mathcal{O},\ g \notin \mathfrak{p}\ \forall\ \mathfrak{p} \in X \}$.

If $\mathfrak{p} \in X$ then $\mathcal{O}_{\mathfrak{p}} = \mathcal{O}(X)_{\underbrace{\mathfrak{p}\mathcal{O}(X)}}$ (same elts in field of fractions K)

localize as $\mathcal{O}(X)$ ideal.

Claim: the unit gp of $\mathcal{O}(X)$ and class gp $Cl(\mathcal{O}(X))$ have similar properties to those of $\mathcal{O} := \mathcal{O}_K$ = ring of integers of K. X: contains almost all places.

Fact 1: $\mathcal{O}_K(X)^*$: unit gp. $\cong \underbrace{\mu(K)}_{\text{rts of 1 in } K} \times \mathbb{Z}^{m+r+s-1}$. r, s as before, m: # of primes not in X

Fact 2: $Cl(\mathcal{O}_K(X))$ is finite. $\longleftarrow$ Note $\mathcal{O}_K(X)$ is Dedekind, so class gp of frac. ideals is well-defined

(can use Minkowski theory to make this quantitative)

~~[illegible]~~ finite set of prime ideals not in X is called S in literature.

then $\mathcal{O}_K(X)$ written $\mathcal{O}_K^S$: "S-integers"

(The multiplicatively closed subset $\mathcal{O}_K \setminus X$ is not same as S as defined above, a set, but in same spirit.)

Both facts are a consequence of the exact sequence

$$1 \to \mathcal{O}_K^\times \hookrightarrow \mathcal{O}(X)^* \to \bigoplus_{\mathfrak{p} \notin X} K^* / \mathcal{O}_{\mathfrak{p}}^* \to Cl(\mathcal{O}) \to Cl(\mathcal{O}(X)) \to 1$$

(modified version of simpler earlier exact sequence.)

+ fact that for ~~[illegible]~~ $K^\times / \mathcal{O}_{\mathfrak{p}}^\times \cong \mathbb{Z}$ so middle piece $\cong \mathbb{Z}^m$

Exactness of the sequence is not too hard. The map to middle given by:

$$\mathcal{O}(X)^\times \hookrightarrow K^\times \longrightarrow \bigoplus_{\mathfrak{p}\notin X} K^\times/\mathcal{O}_\mathfrak{p}^\times$$

$$a \longmapsto a \longmapsto (a \bmod \mathcal{O}_\mathfrak{p}^\times, \dots)$$

If $a \in \mathcal{O}(X)^\times$ is in kernel of this map, then $a \in \mathcal{O}_\mathfrak{p}^\times$ for $\mathfrak{p}\notin X$.

For $\mathfrak{p}\in X$, we know $\mathcal{O}_\mathfrak{p} = \mathcal{O}(X)_{\mathfrak{p}\cdot\mathcal{O}(X)} \supseteq \mathcal{O}(X) \ni a$ so $a \in \mathcal{O}_\mathfrak{p}^\times$ for $\mathfrak{p}\in X$

But we knew from previous prop. that $\bigcap_{\mathfrak{p}:\text{prime}} \mathcal{O}_\mathfrak{p}^\times = \mathcal{O}^*$

so this gives exactness at $\mathcal{O}(X)^*$.

For the next map, we must produce ideal from elts of $\bigoplus_{\mathfrak{p}\notin X} K^\times/\mathcal{O}_\mathfrak{p}^\times$

$$\underbrace{(\dots, \underset{\bmod \mathcal{O}_\mathfrak{p}^\times}{\alpha_\mathfrak{p}}, \dots)}_{\mathfrak{p}\notin X} \longmapsto \prod_{\mathfrak{p}\notin X} \mathfrak{p}^{v_\mathfrak{p}(\alpha_\mathfrak{p})}$$

, then take ideal class of resulting ideal.

with $v_\mathfrak{p}$: valuation from $\mathcal{O}_\mathfrak{p}$.

i.e. $\alpha_\mathfrak{p} \in K^\times$ as $\mathcal{O}_K$ ideal will have factorization, take exponent of $\mathfrak{p}$.

By the way, this valuation $v_\mathfrak{p} : K^\times \longrightarrow \mathbb{Z}$ has kernel $\mathcal{O}_\mathfrak{p}^\times$ giving $K^\times/\mathcal{O}_\mathfrak{p}^\times \simeq \mathbb{Z}$

Again, given elt. $(\alpha_\mathfrak{p})_{\mathfrak{p}\notin X}$ of kernel, this means

$$(\alpha_\mathfrak{p})_{\mathfrak{p}\notin X} \mapsto \prod_{\mathfrak{p}\notin X} \mathfrak{p}^{v_\mathfrak{p}(\alpha_\mathfrak{p})} = (\alpha) \quad \text{i.e. is principal.}$$

By unique factorization, this must be prime decomposition of (α).

$$\Rightarrow \quad v_\mathfrak{p}(\alpha) = 0 \ \forall\, \mathfrak{p}\in X, \quad v_\mathfrak{p}(\alpha) = v_\mathfrak{p}(\alpha_\mathfrak{p}) \ \forall\, \mathfrak{p}\notin X.$$

But these are precisely the statements needed for exactness here at $\bigoplus_{\mathfrak{p}} K^*/\mathcal{O}_{\mathfrak{p}}^\times$ since the first implies $\alpha \in \bigcap_{\mathfrak{p} \in X} \mathcal{O}_{\mathfrak{p}}^\times = \mathcal{O}(X)^*$, and the second implies $\alpha \equiv \alpha_p \mod \mathcal{O}_{\mathfrak{p}}^\times \ \forall \ \mathfrak{p} \notin X$.

Finally the last map $Cl(\mathcal{O}) \to Cl(\mathcal{O}(X))$ is induced from our earlier correspondence on ideals $\mathfrak{a} \mapsto \mathfrak{a}\,\mathcal{O}(X)$, in particular taking primes $\mathfrak{p}$ in X to (all) prime ideals of $\mathcal{O}(X)$. These generate $Cl(\mathcal{O}(X))$ (previously we argued every ideal class contains integral ideal of bounded absolute norm, but we could have taken this one step further, using unique factorization.)

So indeed this map is surjective. What about the kernel?

$\mathfrak{p} \cdot \mathcal{O}(X) = (1) \iff$ prime ideals $\mathfrak{p} \notin X$. So then kernel is any ideal of form $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ with $\mathfrak{p}_i \notin X$, the precise image of the map from $\bigoplus_{\mathfrak{p}} K^*/\mathcal{O}_{\mathfrak{p}}^\times$. //

Nice properties of ring of integers: $\mathcal{O}_K$: integrally closed, Noetherian domain, all primes maximal, has integral basis

When we <u>localize</u>: Preserve everything but integral basis

When we <u>examine orders</u>: inside $\mathcal{O}_K$ so Noetherian, all primes maximal (e.g. primes $\mathfrak{p}$ have full rank so $\mathcal{O}/\mathfrak{p}$ is fin. dim.), and has integral basis by definition. but not <u>integrally</u> closed.

Give up both properties: "one-dimensional Noetherian domains" → turn out to have interesting geometry... as in Krull dimension. (proper chains of prime ideals)

Can we prove similar results using similar exact sequence for 1-dim'l Noetherian domains? Yes, with a little care.

Since no longer integrally closed, the set of all fractional ideals is no longer a group. (pf used linear algebra to show $\mathfrak{p}^{-1} = \{x \in K \mid x \cdot \mathfrak{p} \subseteq \mathcal{O}\}$ with $\mathfrak{a}\mathfrak{p}^{-1} \neq \mathfrak{a}$.)

Fix: Consider the set of invertible fractional ideals. Those ideals $\mathfrak{a}$ s.t. $\exists$ ideal "$\mathfrak{a}^{-1}$" with $\mathfrak{a} \cdot \mathfrak{a}^{-1} = \mathcal{O}$

This set is an abelian gp, under mult. of ideals. Inverses still characterized by: $\mathfrak{a}^{-1} = \{ x \in K \mid x \cdot \mathfrak{a} \subseteq \mathcal{O} \}$ since this is the largest ideal $\mathfrak{b}$ with $\mathfrak{a}\mathfrak{b} \subseteq \mathcal{O}$. Call the gp. $J(\mathcal{O})$ (before wrote J_K for $J(\mathcal{O}_K)$)

H contains principal fractional ideals, $\alpha \cdot \mathcal{O}$ with $\alpha \in K$, called $P(\mathcal{O})$.

Define class gp. of 1-dim'l Noetherian domain as $\mathrm{Pic}(\mathcal{O}) = J(\mathcal{O})/P(\mathcal{O})$. "Picard gp"

Compare with Picard gp. of curve C (proj. variety of dim. 1) over alg. closed field $\overline{K}$,

then have gp. of divisors: formal $\mathbb{Z}$-linear sums of points on C. ($\mathrm{Div}(C)$)

C smooth: If $f \in \overline{K}(C)^{*}$, $\mathrm{div}(f) = \sum_{P \in C} \mathrm{ord}_P(f) P$. D is principal if $D = \mathrm{div}(f)$ some f. call it $P(C)$

then $\mathrm{Pic}(C) := \mathrm{Div}(C)/P(C)$.