On Monday, finished with discussion of 1-dim'l Noetherian domains: $\mathcal{O}$

(common generalization of localizations of Dedekind domains and of orders)

Fractional ideals in $\mathcal{O}$ not a group, in general, so consider set of invertible (fractional) ideals, $J(\mathcal{O})$, which <u>is</u> a group under mult. of ideals.

Knowing inverses exist, we can still characterize inverse of ideal $\mathfrak{a}$ in same way:

$$\mathfrak{a}^{-1} = \{ x \in K \mid x \cdot \mathfrak{a} \subseteq \mathcal{O} \}$$

because this is the largest ideal among $\mathfrak{b}$ such that $\mathfrak{a} \cdot \mathfrak{b} \subseteq \mathcal{O}$. $\leftarrow$ that property characterizes the inverse, if one exists.

Example of <u>how this</u> fails: Pick order not equal to $\mathcal{O}_K$ (so not integrally closed)

then e.g. $\mathcal{O} = \mathbb{Z}[\sqrt{5}] \subseteq \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$ in $K = \mathbb{Q}(\sqrt{5})$.

Know that principal fractional ideals are invertible since, for $\alpha \in K^\times$, consider $\left(\frac{1}{\alpha}\right)$ as principal fractional ideal.

But $(2) \subsetneq (2, 1+\sqrt{5})$, a maximal ideal in order. (calculate discriminants)

And $(2, 1+\sqrt{5})^2 = (2)(2, 1+\sqrt{5})$ so $(2, 1+\sqrt{5})$ is not invertible.

<u>Another characterization</u>: A fractional ideal $\mathfrak{a}$ in $\mathcal{O}$ is invertible $\iff$ $\forall$ primes $\mathfrak{p} \neq 0$, $\mathfrak{a}_\mathfrak{p} := \mathfrak{a} \cdot \mathcal{O}_\mathfrak{p}$ is a principle fractional ideal.

proof: $(\Rightarrow)$ If $\mathfrak{a}$ invertible then $\exists$ ideal $\mathfrak{b}$ with $\mathfrak{a}\mathfrak{b} = \mathcal{O}$

$\Rightarrow$ $\exists$ a_i's, b_i's s.t. $\sum_i a_i b_i = 1$, so not all

$a_i b_i$ can be in $\mathfrak{p}\mathcal{O}_\mathfrak{p}$ (after inclusion in $\mathcal{O}_\mathfrak{p}$ for any prime $\mathfrak{p}$)

since $1 \notin \mathfrak{p}\cdot\mathcal{O}_\mathfrak{p}$. This means one of the products $a_i b_i$ is a unit in $\mathcal{O}_\mathfrak{p}$.

claim: $\mathfrak{a}\cdot\mathcal{O}_\mathfrak{p} = (a_i)\cdot\mathcal{O}_\mathfrak{p}$ ($\supseteq$ clear. show $\subseteq$)

if $x \in \mathfrak{a}\mathcal{O}_\mathfrak{p}$ then $x\cdot b_i \in \mathcal{O}_\mathfrak{p}$ $\Rightarrow$ $x = x\cdot b_i \underbrace{(a_i b_i)^{-1}}_{\text{a unit}} \cdot a_i \in (a_i)\mathcal{O}_\mathfrak{p}$

$(\Leftarrow)$ Suppose $\mathfrak{a}\mathcal{O}_\mathfrak{p}$ is principal, say $\alpha_\mathfrak{p}\mathcal{O}_\mathfrak{p}$, $\forall$ primes $\mathfrak{p}$.

We may take these $\alpha_\mathfrak{p}$ in $\mathcal{O}$ (i.e. in $\mathfrak{a}$). Then we claim that

the construct $\mathfrak{a}^{-1} = \{x \in K \mid x\mathfrak{a} \subseteq \mathcal{O}\}$ is indeed an inverse for $\mathfrak{a}$.

If not, then $\exists$ prime (i.e. max'l since $\mathcal{O}$ 1-dim'l) ideal $\mathfrak{p}$ with $\mathfrak{a}\mathfrak{a}^{-1} \subseteq \mathfrak{p} \subseteq \mathcal{O}$

$\mathcal{O}$ Noetherian, so let $a_1, \dots, a_n$ be generators of $\mathfrak{a}$, so each $a_i = \alpha_\mathfrak{p} \cdot \frac{b_i}{s_i}$

$b_i \in \mathcal{O}$, $s_i \in \mathcal{O}\setminus\mathfrak{p}$

$\Rightarrow$ $s_i a_i \in \alpha_\mathfrak{p}\cdot\mathcal{O}$ $\Rightarrow$ $(s_1 \cdots s_n)\cdot a_i \in \alpha_\mathfrak{p}\mathcal{O}$ $\forall\, i = 1, \dots, n$.

$\Rightarrow$ $s_1 \cdots s_n \alpha_\mathfrak{p}^{-1} \cdot \mathfrak{a} \subseteq \mathcal{O}$

$\Rightarrow$ $s_1 \cdots s_n \alpha_\mathfrak{p}^{-1} \subseteq \mathfrak{a}^{-1}$

$\Rightarrow$ $s_1 \cdots s_n = s_1 \cdots s_n \alpha_\mathfrak{p}^{-1}\alpha_\mathfrak{p} \in \mathfrak{a}\mathfrak{a}^{-1} \subseteq \mathfrak{p}$ $\lightning$ since $S = \mathcal{O}\setminus\mathfrak{p}$ is mult. closed.

Essentially, we've shown $J(\mathcal{O})$: group of invertible fractional ideals in $\mathrm{Frac}(\mathcal{O})$ is

isomorphic to $\bigoplus_{\mathfrak{p}} P(\mathcal{O}_{\mathfrak{p}})$ under homomorphism:

$$J(\mathcal{O}) \longrightarrow \bigoplus_{\mathfrak{p}} P(\mathcal{O}_{\mathfrak{p}})$$

[Here P denotes principal fractional ideals, and $\bigoplus$ means all but finitely many entries are $\mathcal{O}_{\mathfrak{p}}$] ← identity elt. w.r.t. gp. operation

$$\mathfrak{a} \longmapsto \mathfrak{a}\mathcal{O}_{\mathfrak{p}} = (\alpha_{\mathfrak{p}}) \leftarrow \text{a principal ideal by previous proposition.}$$

At least, we have a well-defined homomorphism $(**)$

injective because, if $\mathfrak{a}\mathcal{O}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}} \ \forall \mathfrak{p}$ $(*)$ then $\mathfrak{a} \subseteq \bigcap_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}} = \mathcal{O}$, as claimed last time

But then $\mathfrak{a} = \mathcal{O}$ because if $\exists\ \mathfrak{p}$ with $\mathfrak{a} \subseteq \mathfrak{p} \subset \mathcal{O}$ then for this prime $\mathfrak{p}$ $\quad \mathfrak{a}\cdot\mathcal{O}_{\mathfrak{p}} \subseteq \mathfrak{p}\cdot\mathcal{O}_{\mathfrak{p}} \neq \mathcal{O}_{\mathfrak{p}}$ contradicting $(*)$.

For surjectivity, given elt. of $\bigoplus_{\mathfrak{p}} P(\mathcal{O}_{\mathfrak{p}})$, of form $\bigoplus_{\mathfrak{p}} (\alpha_{\mathfrak{p}})$, claim

that $\quad \mathfrak{a} := \bigcap_{\mathfrak{p}} \alpha_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}$ is the fractional ideal of $J(\mathcal{O})$ mapping to this elt.

It is fractional because, since $\alpha_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$ for almost all $\mathfrak{p}$,

$\exists\, c \in \mathcal{O},\ c \neq 0$, s.t. $c\cdot\alpha_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}} \ \forall \mathfrak{p} \Rightarrow$

$$c\mathfrak{a} \subseteq \bigcap_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}} = \mathcal{O} \text{ so fractional.}$$

Now show $\mathfrak{a}\mathcal{O}_{\mathfrak{p}} = \alpha_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}} \ \forall \mathfrak{p}$ (hence maps to $(\alpha_{\mathfrak{p}})_{\mathfrak{p}}$ and is invertible)

"$\subseteq$" easy, "$\supseteq$" uses C.R.T. for 1-dim'l Noetherian domains:

$$\mathcal{O}/\mathfrak{a} \cong \bigoplus_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}/\mathfrak{a}\mathcal{O}_{\mathfrak{p}}$$

~~or use normalization to get Dedekind domain~~

~~since $\mathfrak{a}$ is f.gen. $\mathcal{O}$-module, pick generator of $\mathfrak{a}$. Any such prime has $a\cdot\mathcal{O} \subseteq \mathfrak{p} \subsetneq \mathcal{O}$~~ [illegible]

$(**)$ Note well-defined because $\mathfrak{a}\mathcal{O}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$ for almost all $\mathfrak{p}$ since only survives in localizations with $\mathfrak{a} \subseteq \mathfrak{p}$, and this only happens for finitely many primes $\mathfrak{p}$ ~~(e.g. $\mathfrak{a}$ is full rank submodule of $\mathcal{O}$, so use classification of modules~~)

? Not nec.

Fact: Given Noetherian domain of dim. 1, $\mathcal{O}$, then its integral closure in $\mathrm{Frac}(\mathcal{O})$, $\tilde{\mathcal{O}}$, is a Dedekind domain.

(Tricky to prove, since $\tilde{\mathcal{O}}$ is not a fin. gen. $\mathcal{O}$-module in general, so Noetherian condition is not automatic for $\tilde{\mathcal{O}}$ (other conditions are))

then we have the following exact sequence: (very reminiscent of earlier exact sequence ...)

$$1 \to \mathcal{O}^\times \to \tilde{\mathcal{O}}^\times \to \bigoplus_{\mathfrak{p}} \tilde{\mathcal{O}}_{\mathfrak{p}}^\times / \mathcal{O}_{\mathfrak{p}}^\times \to \mathrm{Pic}(\mathcal{O}) \to \mathrm{Pic}(\tilde{\mathcal{O}}) \to 1$$

In middle, claim $\tilde{\mathcal{O}}_{\mathfrak{p}}^\times / \mathcal{O}_{\mathfrak{p}}^\times$ is trivial for almost all $\mathfrak{p}$ since $\mathcal{O}_{\mathfrak{p}}$ is integrally closed for almost all $\mathfrak{p}$. Again we define "conductor" $\mathfrak{f}$ which contains primes for which quotient $\tilde{\mathcal{O}}_{\mathfrak{p}}^\times / \mathcal{O}_{\mathfrak{p}}^\times$ is non-trivial:

$$\mathfrak{f} := \{ a \in \tilde{\mathcal{O}} \mid a \cdot \tilde{\mathcal{O}} \subseteq \mathcal{O} \}$$ i.e. biggest ideal of $\tilde{\mathcal{O}}$ contained in $\mathcal{O}$.

(non-zero since $\tilde{\mathcal{O}}$ is fin. gen. $\mathcal{O}$-module)

To prove exactness: Write $\mathrm{Pic}(\mathcal{O}) = J(\mathcal{O}) / P(\mathcal{O}) = \bigoplus_{\mathfrak{p}} P(\mathcal{O}_{\mathfrak{p}}) / P(\mathcal{O})$ and $\mathrm{Pic}(\tilde{\mathcal{O}})$ similarly. (takes some thinking. Why not $\tilde{\mathfrak{p}}$: primes of $\tilde{\mathcal{O}}$?)

then we have commutative diagram of exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & K^\times/\mathcal{O}^\times \arrow[r] \arrow[d] & \bigoplus_{\mathfrak{p}} K^\times/\mathcal{O}_{\mathfrak{p}}^\times \arrow[r] \arrow[d] & \mathrm{Pic}(\mathcal{O}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K^\times/\tilde{\mathcal{O}}^\times \arrow[r] & \bigoplus_{\mathfrak{p}} K^\times/\tilde{\mathcal{O}}_{\mathfrak{p}}^\times \arrow[r] & \mathrm{Pic}(\tilde{\mathcal{O}}) \arrow[r] & 1
\end{tikzcd}$$

with $P(\mathcal{O}_{\mathfrak{p}}) \simeq K^\times/\mathcal{O}_{\mathfrak{p}}^\times$, $P(\mathcal{O}) \simeq K^\times/\mathcal{O}^\times$

and the exactness of the desired sequence is the snake lemma

Finally, we can conclude $\mathcal{O}$ shares same properties as for Dedekind domains when $\mathcal{O}$ just <u>order</u> so that $\tilde{\mathcal{O}}$: its normalization is equal to $\mathcal{O}_K$.
ring of ints.

$$\# \operatorname{Pic}(\mathcal{O}) = \frac{h_K}{[\mathcal{O}_K^\times : \mathcal{O}^\times]} \cdot \frac{\# (\mathcal{O}_K/\mathfrak{f})^\times}{\# (\mathcal{O}/\mathfrak{f})^\times}$$

with h_K : class # of K

$\operatorname{rank}(\mathcal{O}_K^\times) = \operatorname{rank}(\mathcal{O}^\times) = r+s-1$.

(pf: use exact sequence, noting $\operatorname{Pic}(\mathcal{O}_K) = Cl(K)$.)

(**) cont. Why is any ideal $\mathfrak{a} \subseteq \mathcal{O}$ only contained in finitely many primes?

Key lemma in pf of unique factorization of Dedekind domains:

Find primes $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ with $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subseteq \mathfrak{a}$ ← classic Noetherian argument

Consider set of ideals failing this criterion. Has maximal elt. ordering by inclusion, ...

But then $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subseteq \mathfrak{a} \subseteq \mathfrak{p} \Rightarrow \mathfrak{p} = \mathfrak{p}_i$ for some i

since $\mathfrak{p}_i$ are maximal and $\mathfrak{p}$ is prime.

Last loose <u>end</u> : pf. that $\mathfrak{p} \nmid \mathfrak{f} \iff \mathcal{O}_\mathfrak{p}$ is integrally closed
"$\mathfrak{p}$ is regular"

where $\mathfrak{f}$: conductor of $\mathcal{O}$ as before.

(In this case $\mathfrak{p} \cdot \tilde{\mathcal{O}}$ is prime in $\tilde{\mathcal{O}}$ with $\mathcal{O}_\mathfrak{p} = \tilde{\mathcal{O}}_{\mathfrak{p}\tilde{\mathcal{O}}}$.)

<u>strategy</u>: $\Rightarrow$ If $\mathfrak{p} \nmid \mathfrak{f}$, let $\tilde{\mathfrak{p}} = \mathfrak{p} \cdot \mathcal{O}_\mathfrak{p} \cap \tilde{\mathcal{O}}$, a prime of $\tilde{\mathcal{O}} \subseteq \mathcal{O}_\mathfrak{p}$

show $\mathcal{O}_\mathfrak{p} = \tilde{\mathcal{O}}_{\tilde{\mathfrak{p}}}$ so $\mathcal{O}_\mathfrak{p}$ is DVR (equal to localization of Dedekind ring $\tilde{\mathcal{O}}$)

$\Leftarrow$ use that $\tilde{\mathcal{O}} \subseteq \mathcal{O}_\mathfrak{p}$ is f. gen. $\mathcal{O}$ module to show $\exists\, s \in \mathcal{O} \setminus \mathfrak{p}$ s.t. $s \cdot \tilde{\mathcal{O}} \subseteq \mathcal{O}$.