Let $\mathcal{O}$: 1-dim'l Noetherian domain, then make $\mathrm{Spec}(\mathcal{O}) =: X$ into topological space by defining closed sets $\{ \mathfrak{p} \text{ prime} \mid \mathfrak{p} \supseteq \mathfrak{a} \}$ for any ideal $\mathfrak{a}$ in $\mathcal{O}$ ①

For applications to arithmetic, too coarse. Consider pair $(X, \mathcal{O}_X)$ where

$\mathcal{O}_X$ is the sheaf of rings given by

$$\mathcal{F} : \underset{\text{open, non-empty}}{\mathcal{U}} \longmapsto \mathcal{O}(\mathcal{U}) = \left\{ \frac{f}{g} \;\middle|\; g \not\equiv 0 \bmod \mathfrak{p} \;\forall \mathfrak{p} \in \mathcal{U} \right\}$$

"structure sheaf on $\mathrm{Spec}(\mathcal{O}) = X$" together with natural ~~[illegible]~~ map

$\mathcal{O}(\mathcal{U}) \longrightarrow \mathcal{O}(V)$ if $V \subseteq \mathcal{U}$ (if $g \not\equiv 0 \bmod \mathfrak{p} \;\forall \mathfrak{p} \in \mathcal{U}$ then $\not\equiv 0 \bmod \mathfrak{p} \;\forall \mathfrak{p} \in V$)

induced by projection $\prod_{\mathfrak{p} \in \mathcal{U}} \mathcal{O}_\mathfrak{p} \to \prod_{\mathfrak{p} \in V} \mathcal{O}_\mathfrak{p}$

Terminology for sheaves:

elements in ring $\mathcal{F}(\mathcal{U})$ are "sections" - def'n of sheaf is that these sections are well behaved with respect to any open covering of open set $\mathcal{U}$.

"stalk" at a point $x \in X$:

$$\mathcal{F}_x := \varinjlim_{\mathcal{U} \ni x} \mathcal{F}(\mathcal{U})$$

so elements of stalk are equivalence classes of sections

$s_\mathcal{U} \sim s_V$ if we can find $W \subseteq \mathcal{U} \cap V$ with $x \in W$

s.t. $s_\mathcal{U}|_W = s_V|_W$ (i.e. apply restriction map to W)

call these "germs" of sections at x.

<u>Fact</u>: stalk of $\mathcal{O}_X$ at $\mathfrak{p}$ is $\mathcal{O}_\mathfrak{p}$.

(follows from definition. $\mathcal{U} = X \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ $\mathfrak{p} \neq \mathfrak{p}_i$ any i.

and $\mathcal{O}_\mathfrak{p} = \left\{ \frac{f}{g} \;\middle|\; g \not\equiv 0 \bmod \mathfrak{p} \right\}$ with natural inclusion $\mathcal{O}(\mathcal{U}) \hookrightarrow \mathcal{O}_\mathfrak{p}$)

Example 1: If $\mathcal{O}$ is DVR, then $\mathrm{Spec}(\mathcal{O}) = \{\mathfrak{m}, (0)\}$
unique max. ideal

$\mathfrak{m}$ - closed pt., (0) - generic point
not closed, its closure is total space X

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so closed sets: $\phi, \{\mathfrak{m}\}, X \Rightarrow$ open sets: $X, (0), \phi$.

and "functions" on $\mathcal{O}$ are elements f with "values" $f \bmod \mathfrak{m}$, $f \in \mathrm{Frac}(\mathcal{O})$

Example 2: If $\mathcal{O}$ is Dedekind domain, $\mathrm{Spec}(\mathcal{O}) = \{\mathfrak{p} \mid \mathfrak{p} \text{ prime}\}$

Now $\mathcal{O}_\mathfrak{p}$ is a DVR with inclusion $\mathcal{O} \hookrightarrow \mathcal{O}_\mathfrak{p}$ with induced map

$$f: X_\mathfrak{p} := \mathrm{Spec}(\mathcal{O}_\mathfrak{p}) \longrightarrow X := \mathrm{Spec}(\mathcal{O})$$

claim: f is morphism of affine schemes. Affine scheme is pair ($X = \mathrm{Spec}(A)$, $\mathcal{O}_X$: structure sheaf) A: ring

Any homom. of rings $\phi: \mathcal{O} \longrightarrow \mathcal{O}'$ induces map on prime ideals

$$f: X' \longrightarrow X, \qquad \mathfrak{p}' \longmapsto \phi^{-1}(\mathfrak{p}')$$

continuous, and corresponding map on $\mathcal{O}(\mathcal{U})$'s, $\mathcal{U}$: open

$$f^*_\mathcal{U}: \mathcal{O}(\mathcal{U}) \longrightarrow \mathcal{O}(\mathcal{U}') \quad \text{where } \mathcal{U}' = f^{-1}(\mathcal{U})$$
$$s \longmapsto s \circ f|_{\mathcal{U}'}$$

"morphism of affine schemes"

with (1) for $V \subseteq \mathcal{U}$ open:

$$\begin{tikzcd} \mathcal{O}(\mathcal{U}) \arrow[r, "f^*_\mathcal{U}"] \arrow[d, "\rho_{\mathcal{U},V}"'] & \mathcal{O}(\mathcal{U}') \arrow[d, "\rho_{\mathcal{U}',V'}"] \\ \mathcal{O}(V) \arrow[r, "f^*_V"'] & \mathcal{O}(V') \end{tikzcd}$$

Not so easy to prove these properties!

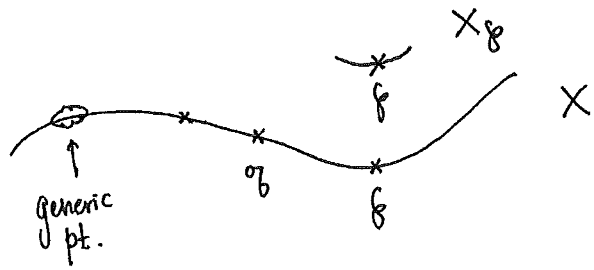
Also can be shown all such morphisms are induced from homs. of rings.

(2) for $\mathfrak{p}' \in \mathcal{U}' \subseteq X'$, and $a \in \mathcal{O}(\mathcal{U})$

$$a(f(\mathfrak{p}')) = 0 \Rightarrow f^*_\mathcal{U}(a) \equiv 0 \mod \mathfrak{p}'.$$

i.e. $a \bmod f(\mathfrak{p}') \equiv 0$

Neukirch's picture:



with stalk at $\mathfrak{p}$ in X

equal to $\mathcal{O}_{\mathfrak{p}}$ — "germ of functions" in infinitesimal nbhd of $\mathfrak{p}$.

The set $\{ \frac{f}{g} \} = \mathcal{O}_{\mathfrak{p}}$ is not defined on nbhd. of $\mathfrak{p}$ in X, which will contain other primes if $\mathcal{O}$ is not itself a local ring.

But any particular $\frac{f}{g}$ in $\mathcal{O}_{\mathfrak{p}}$ has nbhd. on which it is defined. (require that $\mathfrak{q} \in U$ s.t. $g \not\equiv 0 \bmod \mathfrak{q}$)

Claim: For an order $\mathcal{O}$, then if $\mathfrak{p}$ regular, so $\mathcal{O}_{\mathfrak{p}}$ DVR then curve non-singular

But if $\mathcal{O}_{\mathfrak{p}}$ not a DVR, where maximal ideal $\mathfrak{p} \cdot \mathcal{O}_{\mathfrak{p}}$ not generated by single elt., then $\mathfrak{p}$ "singular"

Better to see from geometric setting, reason back to algebraic setting.

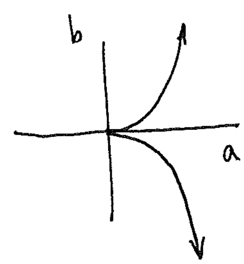
$\mathbb{C}[x]$, $\mathbb{C}[x,y] / y^2 = x^3 + x$ are smooth, but $\mathbb{C}[x,y] / y^2 = x^3 + x^2$ or $y^2 = x^3$ are singular.

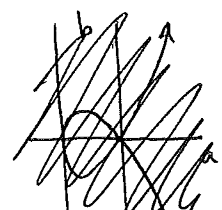
Remember points on these varieties are max'l ideals containing I: quotient ideal.

So $\mathbb{C}[x]$: $(x-a) \longleftrightarrow a \in \mathbb{C}$

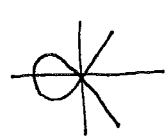
$\mathbb{C}[x,y]/_E$: $(x-a, y-b)$ mod $E: y^2 = f(x) \longleftrightarrow (a,b) \in \mathbb{C}^2$ s.t. $b^2 = f(a)$

Draw real locus: say of $b^2 = a^3$





or $b^2 = a^3 + a^2$



To understand when these varieties are singular, Hartshorne would say

analyze $\mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m}$: maximal ideal of $\mathcal{O}_{\mathfrak{p}}$ in localization $\mathcal{O}_{\mathfrak{p}}$

More precisely, we compute dimension of $\mathfrak{m}/\mathfrak{m}^2$ as $\mathcal{O}_{\mathfrak{p}}/\mathfrak{m}$ - vector space

Then $\mathcal{O}$ is "non-singular" at $\mathfrak{p}$ if $\dim_{\mathcal{O}_{\mathfrak{p}}/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = \dim(\mathcal{O}) = 1$.

(Atiyah-Macdonald tell us that, as a consequence of Nakayama's lemma, if M is $\mathcal{O}_{\mathfrak{p}}$ module, $\mathcal{O}_{\mathfrak{p}}$ local ring, if x_i are basis for $M/\mathfrak{m}M$ as $\mathcal{O}_{\mathfrak{p}}/\mathfrak{m}$ ~~module~~ - vector space then x_i generate M. So suffices to analyze $\mathfrak{m}/\mathfrak{m}^2$ to find gens. for $\mathfrak{m}$.) taking $\mathfrak{m} = M$ in above statement.

Look at the point $(0,0)$ in our three examples:

For each, considering ideal $(x-0, y-0)$ in $\mathbb{C}[x,y]/E$.

$\mathfrak{m}^2 : \langle x^2, y^2, xy \rangle$ so $x \equiv y^2 - x^3$ in $E: y^2 = x^3 + x$

$\equiv 0 \mod \mathfrak{m}^2$

so $\mathfrak{m}/\mathfrak{m}^2$ generated by y.

For other two examples, no relation mod $\mathfrak{m}^2$ on x or y.

But say $(1,1)$ is non-singular on $y^2 = x^3$

since $y - 1 = \frac{1}{2}(x-1)^3 + \frac{3}{2}(x-1)^2 - \frac{1}{2}(y-1)^2 + \frac{3}{2}(x-1) \;/ E$

$\equiv \frac{3}{2}(x-1) \mod \mathfrak{m}^2$