Example : $K = \mathbb{F}_p(t)$ itself, scheme obtained from pair of affine schemes $\mathcal{U} = \mathrm{Spec}(\mathbb{F}_p[u])$, $V = \mathrm{Spec}(\mathbb{F}_p[v])$

if we remove the ideal $(u-0)$ from $\mathcal{U}$, then result is

$$\mathcal{U} - (u) = \mathrm{Spec}(\mathbb{F}_p[u, u^{-1}]), \quad V - (v) = \mathrm{Spec}(\mathbb{F}_p[v, v^{-1}])$$

Call $u = \mathfrak{p}_0$, $v = \mathfrak{p}_\infty$ (pre-saging relation to projective space)

Isomorphism of rings $\mathbb{F}_p[u, u^{-1}] \longrightarrow \mathbb{F}_p[v, v^{-1}]$ yields

$$f : u \longmapsto v^{-1}$$

isomorphism of affine schemes

$$V - (v) \longrightarrow \mathcal{U} - (u)$$

$$\mathfrak{p} \longmapsto f^{-1}(\mathfrak{p})$$

form scheme by identifying $V - (v)$ and $\mathcal{U} - (u)$ in $\mathcal{U} \cup V$.

gives top. space $X = \mathcal{U} \cup V /_{\sim}$ with sheaf of rings $\mathcal{O}_X$ from $\mathcal{O}_{\mathcal{U}}, \mathcal{O}_V$.

p-adic numbers - Hensel's pt. of view

Taylor expansion of $f(z) = \frac{g(z)}{h(z)} \in \mathbb{C}(z)$ has expansion at $z=a$ in non-neg. powers of $(z-a)$, provided a is not a root of $h(z)$.

In algebraic geometry terms, saying $f \in \mathbb{C}[z]_{\mathfrak{p}}$ has expansion in powers of $\mathfrak{p} = (z-a)$. (though we may need to work in a completion for nice class of power series to converge) w/ coeffs in residue class field

Same holds for $f \in \mathbb{Z}_{(p)}$ (and its generalizations)

clear that every positive integer has expansion $f = a_0 + p\cdot a_1 + \cdots + p^n \cdot a_n$ some n, a_i's primitive residues $\{0,\ldots,p-1\}$

(take remainders a_i upon successive division by p)

How do we represent (-1) in p-adic expansion?

That is, let $\mathbb{Z}_p$ "p-adic integers", formal infinite series $\sum_{i=0}^{\infty} a_i p^i := \lim_{N\to\infty} \sum_{i=0}^{N} a_i p^i$ $a_i \in \{0,\ldots,p-1\}$

formally, just a sequence of partial sums S_N

<u>claim</u>: Every elt. of $\mathbb{Z}_{(p)}$ has p-adic expansion (i.e. is in $\mathbb{Z}_p$)

pf: Same idea as above: Let $\bar{S}_N = f \bmod p^{N+1}$ (in $\mathbb{Z}/p^{N+1}\mathbb{Z}$)

So $\bar{S}_0 = f \bmod p \equiv a_0 \in \{0,\ldots,p-1\}$

$\bar{S}_1 = f \bmod p^2 \equiv a_0 + a_1 p$ (with a_0 same as above)

(using that $a_0 + a_1 p$, $a_0, a_1 \in \{0,\ldots,p-1\}$ parametrize residue classes in $\mathbb{Z}/p^2\mathbb{Z}$) and similarly for higher powers of p.

so -1 has p-adic expansion with $a_i = p-1 \ \forall\, i$.

Now extend expansions (as with Laurent series) to include finitely many negative exponents. Call resulting expansions $\mathbb{Q}_p$ "p-adic numbers"

Any elt. in $\mathrm{Frac}(\mathbb{Z}) = \mathbb{Q}$ representable as p-adic number:

write as $\underbrace{\frac{g}{h}}_{f} \cdot p^{-m}$ with $m \in \mathbb{Z}$, $(gh, p) = 1$.

then f has p-adic expansion with lowest term p^{-m}. So consider $\mathbb{Q} \leq \mathbb{Q}_p$ taking $\mathbb{Z} \leq \mathbb{Z}_p$ via this correspondence

Of course, picking primitive residues considered as elts of $\mathbb{Z}$ or of $\mathbb{Q}$ was rather arbitrary. Better to just consider expansion as infinite sequence s_N taking values in $\mathbb{Z}/p^{N+1}\mathbb{Z}$.

with canonical projections $\pi_N : \mathbb{Z}/p^{N+1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^N\mathbb{Z}$

taking s_N to s_{N-1}.

That is, elements lie in projective limit of rings $\mathbb{Z}/p^N\mathbb{Z}$ with restriction maps π_N

denoted $\varprojlim_N \mathbb{Z}/p^N\mathbb{Z}$. Elements are represented

by infinite sequence of elts s_N with values in $\mathbb{Z}/p^{N+1}\mathbb{Z}$.

This establishes bijection

$$\mathbb{Z}_p \xrightarrow{\sim} \varprojlim_N \mathbb{Z}/p^N\mathbb{Z}$$

formal p-adic integers

Since $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ has natural ring structure, we obtain one on $\mathbb{Z}_p$ via identification. To put similar structure on $\mathbb{Q}_p$, writing any $f \in \mathbb{Q}_p$ as $f = p^{-m} g$ with $g \in \mathbb{Z}_p$, which shows how to extend addition/mult. from $\mathbb{Z}_p$ to $\mathbb{Q}_p$, making $\mathbb{Q}_p$ the field of fractions of $\mathbb{Z}_p$.

Final algebraic result: Given polynomial eq'n with integer coeffs, p: prime $F(x_1, \dots, x_n)$

then $F(x_1, \dots, x_n) \equiv 0 \mod p^\nu$ is solvable for all $\nu \geq 1$

$\Leftrightarrow$ $F(x_1, \dots, x_n) = 0$ is solvable in $\mathbb{Z}_p$.

Pf: ($\Leftarrow$) $F = 0$ over $\mathbb{Z}_p$, restricting to components in $\mathbb{Z}/p^\nu\mathbb{Z}$, becomes $F \equiv 0$ (mod p^ν)

and so if $(x_1, \dots, x_n)$ is sol'n in $\mathbb{Z}_p^n$, its components at ν are sol'ns mod p^ν.

($\Rightarrow$) Problem here is that we need to show $\exists$ sol'n in $\varprojlim_\nu \mathbb{Z}/p^\nu\mathbb{Z}$ not just $\prod_{\nu=1}^\infty \mathbb{Z}/p^\nu\mathbb{Z}$. No harder in n variables than in 1 variable, so just consider $F(x) \equiv 0 \pmod{p^\nu}$ with solns x_ν for each ν.

Plan is to pick subsequence of $\{x_\nu\}_{\nu \in \mathbb{N}}$ in projective limit.

By pigeonhole principle, $\exists$ residue class mod p y_1 with infinitely x_ν s.t. $x_\nu \equiv y_1 \ (p)$

That subsequence will have both $x_0 \equiv y_1 \ (p)$ and $F(x_0) \equiv 0 \ (p)$

repeat with y_2 mod p^2 where clear that $y_2 \equiv y_1 \ (p)$ since all elements in subsequence have this property. The resulting p-adic integer $y = (y_k)$ is desired sol'n in $\mathbb{Z}_p$.

Analytic reason for its importance:

$\mathbb{Q}_p$ is completion of $\mathbb{Q}$ with respect to $|\cdot|_p$: p-adic valuation.

the natural valuation on $\mathbb{Z}_{(p)}$ extended to $\mathbb{Q}$ by

writing $a \in \mathbb{Q} = p^m \cdot \frac{b}{c}$ with $\gcd(bc, p) = 1$, $m \in \mathbb{Z}$

then $|a|_p = p^{-m}$. This way $|0|$ "=" $p^{-\infty} = 0$.

$= p^{-v_p(a)}$ i.e. $v_p(a) = m = \infty$.

and v_p a valuation with usual props $\Rightarrow$ $|\cdot|_p$ is a norm.

We want to show that, up to some equivalence, $|\cdot|_p$ and usual absolute value $|\cdot| =: |\cdot|_\infty$ exhaust all possible norms on $\mathbb{Q}$:

(1) $|a|_p = 0 \iff a = 0$

(2) $|ab|_p = |a|_p |b|_p$

this is requirement

(3) $|a+b|_p \leq \max\{|a|_p, |b|_p\} \leq |a|_p + |b|_p$

equivalence: any other norm is a real power of one of these

$|\cdot| = |\cdot|_p^s$ $s \in \mathbb{R}$.
p: prime or $\{\infty\}$.

To be proved:
$\mathbb{Q}_p$ is complete w.r.t. $|\cdot|_p$

One way to see that we've obtained all valuations is via product formula.

Proposition: For all $a \in \mathbb{Q}$, $a \neq 0$, then

$$\prod_{v:\mathrm{val}} |a|_v = 1.$$

(check $\prod_{p<\infty} |a|_p = \pm a^{-1} = \frac{a}{|a|_\infty} \cdot a^{-1}$

i.e. $a = \frac{a}{|a|_\infty} \prod_{p<\infty} \frac{1}{|a|_p}$)

$\Rightarrow$ result.

In function field, also have discrete valuation

$|\cdot|_{\mathfrak{p}}$ corresponding to primes $\mathfrak{p} = p(t)$, monic irred. poly.

Write $f(t) = p(t)^m \frac{g(t)}{h(t)}$ $(g \cdot h, p) = 1$.

then $|f(t)|_{p(t)} = q^{-m} = q^{-\mathrm{val}_{p(t)}(f(t))}$ where $q = p^{\deg(p(t))}$

+ degree valuation $|\cdot|_\infty$ where $|f|_\infty = q^{-v_\infty(f)} = q^{-(\deg(h) - \deg(g))}$

if $f = \frac{g}{h}$.

and again product formula holds.

$f = \frac{g}{h} = \frac{p_1(t)^{e_1} \cdots p_r(t)^{e_r}}{\tilde{p}_1(t)^{\tilde{e}_1} \cdots \tilde{p}_s(t)^{\tilde{e}_s}}$ with difference in powers of q made up by $|\cdot|_\infty$.