Two ways of understanding $\mathbb{Q}_p$:

$\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$ with $\mathbb{Z}_p := \varprojlim_N \mathbb{Z}/p^N\mathbb{Z}$ OR

As completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ where $|a|_p := p^{-v_p(a)}$

where $v_p(a)$: valuation – behaves well with respect to addition / multiplication –

defined by writing $a = p^m \frac{b}{c}$ with $\gcd(bc, p) = 1$, $m \in \mathbb{Z}$

this implies $|a+b|_p \leq$ max's $\leq$ sum

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Let R denote the ring of all Cauchy sequences in $|\cdot|_p$.

component-wise addition + mult.

Note formal power series $\sum_i a_i p^i$, $a_i \in \{0, \dots, p-1\}$ are all

Cauchy because $|x_n - x_m|_p = |\sum_{i=m}^{n-1} a_i p^i|_p \leq \max_{m \leq i < n} \{|a_i p^i|_p\} \leq p^{-m}$

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Let $\mathfrak{M}$ denote subset of Cauchy sequences converging to 0. A (maximal) ideal in R.

then let $\mathbb{Q}_p := R/\mathfrak{M}$ (of course, we'll need to show this def'n of $\mathbb{Q}_p$ agrees with old one.)

How to embed $\mathbb{Q}$ in $\mathbb{Q}_p$?

$a \mapsto (a, a, \dots)$. Previously, we mapped a to partial sums $\bar{s}_N$ in $\mathbb{Z}/p^{N+1}\mathbb{Z}$

e.g. an integer like 35 in $\mathbb{Q}_3$ would have partial sums (35 (mod 3), 35 (mod 9), ...)

$= (2, 8, 8, 35, 35, \dots)$

$\sim (35, 35, \dots)$ in $R/\mathfrak{M}$.

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Define $|x|_p$ for $x = \{x_n\} \in R/\mathfrak{M}$

to be $|x|_p := \lim_{n \to \infty} |x_n|_p \in \mathbb{R}$.

($\mathbb{R}$ complete, $|x_n|_p$ Cauchy, so limit exists)

Well-defined since elts of $\mathfrak{M}$ have limit 0.

Similarly, we can extend the valuation to $\mathbb{Q}_p$ by setting

$$v_p(x_n) = -\log_p |x_n|_p \quad \text{and} \quad v_p(x) = \lim_{n\to\infty} v_p(x_n)$$

$$x = \{x_n\}$$

Since valuation is discrete, only possibilities

are $v_p(x) = \infty$ or $\{v_p(x_n)\}$ is Cauchy, values in $\mathbb{Z}$, so

$$v_p(x) = v_p(x_n) \ \forall\, n \geq n_0$$

some n_0.

then remains true that with $x = \{x_n\}$,

$$|x|_p = p^{-v_p(x)}.$$

With these def'ns, mimic proof that $\mathbb{R}$ is complete w.r.t. $|\cdot|_\infty$ on $\mathbb{Q}$.

i.e. every Cauchy sequence $\{x_n\}$ converges in $\mathbb{Q}_p$.

of equivalence classes of Cauchy sequences

So $\{a_j\}_j$ where each $a_j = \{x_i\}$ so could use double index $\{a_{ji}\}$

Find N_j for each j such that if $i, i' \geq N_j$ then $|a_{ji} - a_{ji'}| < p^{-j}$

Then $\{a_{j,N_j}\}_{j=1,\dots}$ is the limit of $\{a_j\}$

Still need to show this def'n of $\mathbb{Q}_p$ matches algebraic one. Continue to postpone this...

Recall that $|\cdot|_p$ satisfies stronger form of triangle inequality:

$$|x+y|_p \leq \max\{|x|_p, |y|_p\}. \quad \text{This implies:}$$

Proposition $\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$

is a subring of $\mathbb{Q}_p$, and the closure of $\mathbb{Z}$ w.r.t. $|\cdot|_p$ in $\mathbb{Q}_p$.

pf: subring follows from add/mult. props. of valuation.

If $\{x_n\}$ Cauchy sequence in $\mathbb{Z} \subseteq R/\mathfrak{m}$ and $x = \lim_{n\to\infty} x_n$

then since $|x_n|_p \le 1$, have $|x|_p \le 1$ so $x \in \mathbb{Z}_p$. For converse,

if $x = \lim_{n\to\infty} x_n \in \mathbb{Z}_p$ with $\{x_n\}$ Cauchy, in $\mathbb{Q}$, then $\exists\, n_0$ s.t. $n \geq n_0$ then

$|x|_p = |x_n|_p \le 1$. For these $x_n = \frac{a_n}{b_n}$ must have $(b_n, p) = 1$

so for each such n, find $y_n \in \mathbb{Z}$ s.t. $y_n \equiv \frac{a_n}{b_n} \mod p^n$, i.e.

$|x_n - y_n|_p = \left|\frac{a_n}{b_n} - y_n\right|_p \le \frac{1}{p^n}$ so $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ with $y_n \in \mathbb{Z}$

so in $\overline{\mathbb{Z}}$.

Corollary: Units of $\mathbb{Z}_p$ ($\mathbb{Z}_p^*$) are $\{x \in \mathbb{Z}_p \mid |x|_p = 1\}$

and any $x \in \mathbb{Q}_p^\times$ admits representation $x = p^m \cdot u$ with $m \in \mathbb{Z}$, $u \in \mathbb{Z}_p^*$

pf: First is clear. If $v_p(x) = m$ then $v_p(xp^{-m}) = 0$ (i.e. $|xp^{-m}|_p = 1$)

Proposition: Non-zero ideals of the ring $\mathbb{Z}_p$ are principal ideals $p^n \mathbb{Z}_p$, $n \geq 0$.

$\{x \in \mathbb{Q}_p \mid v_p(x) \geq n\}$. So $\mathbb{Z}_p$ is DVR.

Moreover $\mathbb{Z}_p / p^n\mathbb{Z}_p \simeq \mathbb{Z}/p^n\mathbb{Z}$

pf: For any non-zero ideal $\mathfrak{a}$, find element with smallest valuation m: $x = p^m u$ $u \in \mathbb{Z}_p^\times$ (know $m \geq 0$ since $x \in \mathbb{Z}_p$). Then $\mathfrak{a} = (x) = p^m \mathbb{Z}_p$. (Any $y = p^n u' \in \mathfrak{a}$ has $n \geq m$.)

For the isomorphisms $\mathbb{Z}/p^n\mathbb{Z} \simeq \mathbb{Z}_p/p^n\cdot\mathbb{Z}_p$, we have natural homom.

$\phi : \mathbb{Z} \longrightarrow \mathbb{Z}_p/p^n\mathbb{Z}_p$

$a \longmapsto a \bmod p^n\mathbb{Z}_p$.

If $x \in \mathbb{Z}_p$, then since $\mathbb{Z}_p = \overline{\mathbb{Z}}$,

we can find $a \in \mathbb{Z}$ with $|x-a| \le \frac{1}{p^n}$

$\Rightarrow x-a \in p^n\mathbb{Z}_p$ so ϕ is surjective

Kernel is $p^n\cdot\mathbb{Z}$.

To link algebraic and analytic definitions, use this isomorphism $\mathbb{Z}/p^n\mathbb{Z} \simeq \mathbb{Z}_p/p^n\mathbb{Z}_p$ ~~answers~~ which gives surjective homoms. $\underbrace{\mathbb{Z}_p}_{\text{analytic}} \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$ and hence homom. $\mathbb{Z}_p \longrightarrow \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$

(i.e. the homoms for each n will give compatible family of homs. w.r.t. canonical projection)

<u>claim</u>: $\mathbb{Z}_p \longrightarrow \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ is isomorphism.

<u>pf</u>: <u>injectivity</u>: if $x \in \mathbb{Z}_p$ mapped to 0, then $x \in p^n\mathbb{Z}_p$ $\forall n$

$\Rightarrow |x|_p \le \frac{1}{p^n}$ $\forall n$ $\Rightarrow |x|_p = 0 \Rightarrow x = 0$. ✓

<u>surjectivity</u>. Elements in proj. limit are sequences of partial sums S_N with

$S_N = \sum_{i=0}^{N} a_i p^i \quad a_i \in \{0,\dots,p-1\}$. This is Cauchy sequence in $\mathbb{Z}_p$ so converges to elt

$x = \sum_{i=0}^{\infty} a_i p^i \in \mathbb{Z}_p$.

Since $x - S_N = \sum_{i=N+1}^{\infty} a_i p^i \in p^{N+1}\mathbb{Z}$

so $x \equiv S_N \bmod p^{N+1}$ $(\forall N)$ i.e. x is mapped to $\{S_N\}_N$. ✓

Final algebraic characterization: $\mathbb{Z}_p \simeq \mathbb{Z}[[x]]/(x-p)$ ~ formal power series w/ coeffs in $\mathbb{Z}$

closer to our earlier geometric formulations. uses natural homom. $\mathbb{Z}[[x]] \to \mathbb{Z}_p$