proof of claim that $|m|^{1/\log m} = |n|^{1/\log n}$ $\forall$ $m,n > 1$, integers.

Analyze function $f(m) := \max(0, \log|m|)$:

- $f(m^k) = k \cdot f(m)$ $\forall$ m,k
- $f(mn) \le f(m) + f(n)$ (since one could give 0) $\forall$ m,n
- $f(m+n) \le \log 2 + \max(f(m), f(n))$ $\forall$ m,n.

Want to show

$$\frac{f(m)}{\log m} = \frac{f(n)}{\log n} \quad \forall\, m,n$$

(follows because $|m+n| \le |m| + |n| \le 2 \cdot \max\{|m|, |n|\}$)

and implies more generally, by induction,

$$f\Big(\sum_{i=0}^{r} m_i\Big) \le r \cdot \log 2 + \max_i \{f(m_i)\} \qquad (*)$$

Write m in n-ary expansion $m = \sum_{i=0}^{r} a_i n^i$ with $n^r \le m < n^{r+1}$, $a_i \in \{0, \dots, n-1\}$ $\forall i$

Let $b = \max\{f(0), \dots, f(n-1)\}$.

Then $f(a_i n^i) \le b + i \cdot f(n)$ $\forall i$, so by $(*)$,

$f(m) \le r \cdot \log 2 + b + r \cdot f(n)$ since $i \le r$.

Now $n^r \le m \iff r \cdot \log n \le \log m$ so

$$\frac{f(m)}{\log m} \le \frac{\log 2 + f(n)}{\log n} + \frac{b}{\log m}.$$

Substitute $m \mapsto m^k$
let $k \to \infty$

LHS: $\dfrac{f(m^k)}{\log m^k} = \dfrac{f(m)}{\log m}$

RHS: $\dfrac{\log 2 + f(n)}{\log n} + \dfrac{b}{k \cdot \log m}$ $\to 0$ as $k \to \infty$

so $\dfrac{f(m)}{\log m} \le \dfrac{a + f(n)}{\log n}$.

now replace $n \mapsto n^k$
take $k \to \infty$:

$f(m)/\log m \le f(n)/\log n$

now reverse roles of m,n to get reverse ineq.

Return to setting of general field, K, and again use dichotomy of valuations – archimedean v. non-arch. – to study them.

If v with assoc. $|\cdot|_v$ is non-archimedean then by 3 axioms for non-arch. valuation

know $\quad \mathcal{O} = \{ x \in K \mid v(x) \geq 0 \} = \{ x \in K \mid |x|_v \leq 1 \}$

is subring of K with units

$\mathcal{O}^{\times} = \{ x \in K \mid |x|_v = 1 \}$ and unique maximal ideal $\mathfrak{p} = \{ x \in K \mid |x|_v < 1 \}$

(Moreover, $\mathcal{O}$ is integral domain (since K is) with field of fractions K)
where either $x \in K$ is in $\mathcal{O}$ or $x^{-1} \in \mathcal{O}$.

"valuation ring"

Fact: $\mathcal{O}$ is integrally closed. (in $\mathrm{Frac}(\mathcal{O}) = K$) Thus if $K = \#$ field, v: valuation (non-arch.), then $\mathbb{Z} \subseteq \mathcal{O}_v$ so $\mathcal{O}_K \subseteq \mathcal{O}_v$

pf: Any elt $x \in K$ integral (over $\mathcal{O}$) satisfies monic equation

$x^n + a_1 x^{n-1} + \cdots + a_n = 0$ with $a_i \in \mathcal{O}$. Want to show $x \in \mathcal{O}$.

If not, then since $\mathcal{O}$ valuation ring, $x^{-1} \in \mathcal{O}$. But then

$$x = -a_1 - a_2 x^{-1} - \cdots - a_n x^{-(n-1)} \in \mathcal{O}. \quad \text{↯}.$$

Examples: $K = \mathbb{Q}$, $v \longleftrightarrow p$: prime. then $\mathcal{O}_v = \mathbb{Z}_{(p)} = \underbrace{\{ \tfrac{a}{b} \mid p \nmid b \}}_{\text{localization at } p.}$

(similarly for $\#$ fields)

$K = \mathbb{Q}_p$, then $\mathcal{O} = \mathbb{Z}_p$.

Say that valuation is "discrete" if it admits smallest positive value m.

Then the set of all possible valuations is $m \cdot \mathbb{Z}$ for elts. of K.

Always find equivalent valuation with $m=1$. ("normalized" valuations)

Note that sets $\mathcal{O}, \mathcal{O}^\times, \mathfrak{p}$ are independent of representative in equivalence class.

Final <u>Proposition</u>: if v is <u>discrete</u> then valuation ring $\mathcal{O}_v$ is P.I.D.

(so $\mathcal{O}_v$ is discrete valuation ring) with $\mathfrak{p}^n / \mathfrak{p}^{n+1} \simeq \mathcal{O}_v / \mathfrak{p}$ $\forall n$.

Moreover the chain of ideals $\mathcal{O} \supseteq \mathfrak{p} \supseteq \mathfrak{p}^2 \supseteq \cdots$ form a basis of open nbhds of 0 in K. $\left(\mathfrak{p}^n = \left\{ x \in K \;\middle|\; |x|_v < \frac{1}{q^{n-1}} \text{ if } |\cdot| = q^{-v_\mathfrak{p}(\cdot)} \right\} \right)$

Similarly, $1 + \mathfrak{p}^n$ give base of nbhds of 1 in $\mathcal{O}^\times$.

Archimedean valuations: Given K: field, any valuation v, form completion $\hat{K}$.

if v: archimedean, not many choices for $\hat{K}$:

Theorem (Ostrowski) K: field, $\hat{K}$: completion w.r.t. archimedean v,

then there is an isomorphism $\sigma: \hat{K} \longrightarrow \mathbb{R}$ or $\mathbb{C}$

such that $|a|_v = |\sigma(a)|_\infty^s$ ∞: arch. on $\mathbb{R}$ or on $\mathbb{C}$. with $s \in (0,1]$.

pf: To have ~~non~~arch. valuation, K must have characteristic 0

(else $\{n\}_{n \in \mathbb{N}}$ is finite, and so bounded in absolute value)

Thus K contains $\mathbb{Q}$. So $|\cdot|_v$ restricted to $\mathbb{Q}$ is $|\cdot|_\infty$ (possibly up to equivalence (mult. by $s \in \mathbb{R}$))

and $\hat{\mathbb{Q}} \subseteq \hat{K}$ with $\hat{\mathbb{Q}} \simeq \mathbb{R}$ identifying $|\cdot|_v$ and $|\cdot|_\infty$.

How can we show that in fact $\hat{K} = \mathbb{R}$ or $\mathbb{C}$? Show every $\xi \in \hat{K}$ satisfies quadratic equation $/\mathbb{R}$. Clever idea:

$$f(z) := \xi^2 - (z+\bar{z})\xi + z\bar{z}.$$

then $|f(z)|_v : \mathbb{C} \longrightarrow \mathbb{R}_{\geq 0}$

Show $\exists\ z_0$ with $f(z_0) = 0$. (i.e. $|f(z_0)|_v = 0$) (for each such $\xi \in K$) Then coefficients $z_0 + \bar{z}_0$ and $z_0\bar{z}_0$ are real, so done.

~~Clearly $\lim_{z \to \infty} |f(z)|_v = \infty$~~ Key fact: $|f(z)|_v$ attains its minimum for some z_0.

(Neukirch argues true since $\lim_{z \to \infty} |f(z)|_v = \infty$.)*

Since the set S of values at which minimum is obtained is nonempty, bounded closed

we pick $z_0 \in S$ with $|z_0|_v \geq |z|_v \ \forall z \in S$.

* - this follows b/c $|\xi^2 - (z+\bar{z})\xi + z\bar{z}|_v \geq \big|\, |\xi^2|_v - |(z+\bar{z})\xi + z\bar{z}|_v \big|_\infty$ reverse triangle inequality + fact that $z+\bar{z}$ and $z\bar{z}$ are real so

$|z+\bar{z}|_v = |z+\bar{z}|_\infty^s$ and $|z\bar{z}|_v = |z\bar{z}|_\infty^s$.

if minimum m of $|f(z)|$ is > 0, derive a contradiction.

The real polynomial $g(x) = x^2 - (z_0 + \bar{z}_0)x + z_0\bar{z}_0 + \epsilon$

with $0 < \epsilon < m$ and roots called $z_1, \bar{z}_1 \in \mathbb{C}$.

then $z_1\bar{z}_1 = z_0\bar{z}_0 + \epsilon$ so $|z_1|_v > |z_0|_v \Rightarrow f(z_1) > m$.

(since z_0 has $|z_0|$ maximal with $f(z_0) = m$)

Now fix $n \in \mathbb{N}$ and consider

$$G_n(x) := [g(x) - \epsilon]^n - (-\epsilon)^n$$

$$= \prod_{i=1}^{2n} (x - \alpha_i) = \prod_{i=1}^{2n} (x - \bar{\alpha}_i)$$

one of α_i is $z_1, \bar{z}_1$. Say $\alpha_1 = z_1$.

$$\Rightarrow G_n(x)^2 = \prod_{i=1}^{2n} (x^2 - (\alpha_i + \bar{\alpha}_i)x + \alpha_i\bar{\alpha}_i)$$

Inserting $\xi \in K$, we get

$$|G_n(\xi)|_v^2 = \prod_{i=1}^{2n} |f(\alpha_i)|_v \geq |f(\alpha_1)| \cdot m^{2n-1}$$

Better z_1.

Now triangle inequality:

$$|G_n(\xi)| \leq |\xi^2 - (z_0 + \bar{z}_0)\xi + z_0\bar{z}_0|_v^n + |-\epsilon|_v^n$$

$$= |f(z_0)|_v^n + |-\epsilon|_v^n = |f(z_0)|_v^n + \epsilon^n$$

real so $|\epsilon|_v = |\epsilon|_\infty$

$$= m^n + \epsilon^n.$$

Together these:

$$\Rightarrow |f(z_1)| \, m^{2n-1} \leq (m^n + \epsilon^n)^2$$

$$\Rightarrow \frac{|f(z_1)|}{m} \leq \left(1 + \left(\frac{\epsilon}{m}\right)^n\right)^2$$

taking limits as $n \to \infty$ gives contradiction so m must be 0.