Another example of Hensel's lemma: $f(x) = x^{p-1} - 1 \in \mathbb{Z}_p[x]$.

This splits into distinct linear factors over $\mathbb{Z}_p / p\mathbb{Z}_p \simeq \mathbb{Z}/p\mathbb{Z}$.

So by repeated uses of Hensel's Lemma, separating off one root at a time, we get same splitting into linear factors over $\mathbb{Z}_p$. So $\mathbb{Z}_p$ contains all $(p-1)^{st}$ roots of unity.

Corollary: $K = \hat{K}$ for non-arch. valuation v. If $f(x) = a_0 + \cdots + a_n x^n$ is irreducible in $K[x]$ with $a_0 a_n \neq 0$, then

$$|f| \overset{\text{def}}{=} \max_i \{ |a_i|_v \} = \max \{ |a_0|_v, |a_n|_v \}$$

so if f monic with $a_0 \in \mathcal{O}_v$, then $f \in \mathcal{O}_v[x]$.

pf: ~~clear denominators~~ divide common factors + clear denominators in f to get polynomial in $\mathcal{O}_v[x]$, call it f still, with $|f| = 1$. Let r be minimal such that $|a_r|_v = 1$.

Thus a_i with $i < r$ have $a_i \equiv 0 \mod \wp$ and so we may write:

$$f(x) \equiv x^r (a_r + a_{r+1} x + \cdots + a_n x^{n-r}) \quad (\operatorname{mod} \wp)$$

If $\max \{ |a_0|, |a_n| \} < 1$ then $0 < r < n$ and the congruence would lift, by Hensel's Lemma, to a factorization of f in $\mathcal{O}_v[x]$ contradicting irreducibility of f.

Q: How to arrange $|f|_v = 1$ if v is not discrete? Not really needed in proof. Just set $|f|_v = m$

Use this in next theorem: Let $K = \hat{K}$ w.r.t valuation $|\cdot|_v$. Then $|\cdot|_v$ may be extended in a unique way to valuation of any algebraic extension $L|K$. The new absolute value on L is given by

$$|\alpha|_L = \sqrt[n]{|N_{L/K}(\alpha)|_v} \quad \text{if } \deg(L/K) \text{ finite, and}$$

$L = \hat{L}$ with respect to $|\cdot|_L$.

pf: Immediately reduce to non-arch. case, since only non-trivial ext'n of arch. abs. value on field is $K = \mathbb{R}$, $L = \mathbb{C}$ as $\mathbb{C}$ is algebraically closed.

Then true that $|\alpha|_{\mathbb{C}} = \sqrt[2]{|N_{\mathbb{C}/\mathbb{R}}(\alpha)|_{\mathbb{R}}}$. since $N_{\mathbb{C}/\mathbb{R}}(z) = z \cdot \bar{z}$.
(product over embeddings in sep. ext'n)

In non-arch. case, with $\deg(L/K)$ finite, $= n$.

Existence: check that given formula defines valuation on L, and restricts to initial valuation on K. Definition of $N_{L/K}$ makes many of these easy:

$\alpha = 0 \iff |\alpha|_L = 0$, $N_{L/K}$ multiplicative so $|\alpha\beta|_L = |\alpha|_L |\beta|_L$

and restricts to valuation on K since, if $\alpha \in K$, $N_{L/K}(\alpha) = \alpha^n$.

Remains to check (strong) triangle inequality:

$$|\alpha + \beta|_L \leq \max\{|\alpha|_L, |\beta|_L\}$$ divide by α or β to get equivalently:

show: if $|\alpha|_L \leq 1$ then $|\alpha + 1|_L \leq 1$.

i.e. if $\sqrt[n]{|N_{L/K}(\alpha)|_v} \leq 1$ then $\sqrt[n]{|N_{L/K}(\alpha+1)|_v} \leq 1$. so remove n^{th} roots.

thus we must show if $N_{L/K}(\alpha) \in \mathcal{O}_v$: valuation ring of v,

then $N_{L/K}(\alpha+1) \in \mathcal{O}_v$.

<u>claim</u> : $\{ \alpha \in L \mid N_{L/K}(\alpha) \in \mathcal{O}_v \}$ = integral closure of $\mathcal{O}_v$ in L, $\overline{\mathcal{O}_v}$

(so it is a ring, and hence $\alpha \in \overline{\mathcal{O}}_v \Rightarrow \alpha + 1 \in \overline{\mathcal{O}}_v$)

" $\supseteq$ " : if $\alpha \in \overline{\mathcal{O}}_v$, proved long ago that $N_{L/K}(\alpha) \in \mathcal{O}_v$.

" $\subseteq$ " : if $\alpha \in L^\times$, $N_{L/K}(\alpha) \in \mathcal{O}_v$, then write min. poly. for α / K

$$f(x) = x^d + a_{d-1}x^{d-1} + \dots + a_0 . \qquad N_{L/K}(\alpha) = \pm a_0^m$$

some $m \leq n$.

But if $\pm a_0^m \in \mathcal{O}_v \Rightarrow |\pm a_0^m|_v \leq 1$

$\Rightarrow |a_0|_v \leq 1 \Rightarrow a_0 \in \mathcal{O}_v$.

Applying our corollary to Hensel's lemma, then $f(x) \in \mathcal{O}_v[x]$, i.e. $\alpha \in \overline{\mathcal{O}}_v$.

(the claim also shows $\overline{\mathcal{O}}_v$ is valuation ring of $|\cdot|_L$.)

<u>Uniqueness</u> : if $|\cdot|_L'$ is another valuation extending $|\cdot|_v$ with valuation ring $\mathcal{O}'$

Show $\overline{\mathcal{O}}_v \subseteq \mathcal{O}'$ (and that this determines their equivalence)

if $\alpha \in \overline{\mathcal{O}}_v \setminus \mathcal{O}'$, then letting $f(x)$ as above be min. poly for α / K

then coeffs a_i of f are in fact in $\mathcal{O}_v$. $\alpha \notin \mathcal{O}'$ implies $\alpha^{-1} \in \mathfrak{p}'$ maximal ideal of $\mathcal{O}'$

But then can invert min poly:

$$1 = -a_1 \alpha^{-1} - \dots - a_d (\alpha^{-1})^d \in \mathfrak{p}' \quad \text{↯}.$$

so we have $\overline{\mathcal{O}}_v \subseteq \mathcal{O}'$, i.e. $|\alpha|_L \leq 1 \Rightarrow |\alpha|_L' \leq 1$ which implies valuations are equivalent (and in fact, therefore equal since they agree on K)