"$\longleftarrow$" Given (non-discrete) locally compact field $K$, how to assoc. valuation?

$K$ is locally compact abelian gp., so has unique up to constant translation invariant measure (Haar measure $\mu$). Given automorphism $\phi$ of $G$ then $\mu \circ \phi$ and $\mu$ must agree up to constant $\in \mathbb{R}_+^\times$ (and importantly, that constant is indep. of choice of Haar measure)

If we take our automorphism to be multiplication in $K^\times$

$a \in K^\times : x \longmapsto ax$ for any fixed $a$,

define $\mathrm{mod}_K(a) = \dfrac{\mu(aX)}{\mu(X)}$ $X$: measurable set. Show $\mathrm{mod}_K(\cdot)$ is absolute value $K^\times \to \mathbb{R}_+^\times$. extend to $K$ by $\mathrm{mod}_K(0) = 0$.

Weil shows either $\exists$ prime $p \in \mathbb{Z}$ s.t.

$\mathrm{mod}_K(p \cdot 1_K) < 1$ "$p$-field" $\rightsquigarrow$ image of $\mathrm{mod}_K$ discrete

or $\underbrace{\text{it}}_{K}$ is an algebra over $\mathbb{R}$ "$\mathbb{R}$-field" $\rightsquigarrow$ $\mathbb{R}$ or $\mathbb{C}$.

See discussion on p.12 of Weil's Basic Number Theory

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Using characterization of non-arch. local fields as complete w.r.t. discrete valuation, finite residue field, classify local fields (Weil does this from other definition as locally compact field)

<u>Theorem</u>: Local fields are finite extensions of $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$

<u>Proof</u>: First show all such fields are local fields: ✓ (extensions of either are complete w.r.t. unique extn from $\underbrace{\mathbb{Q}_p \text{ or } \mathbb{F}_p((t))}_{K}$ given by

$|\cdot|_L = \sqrt[n]{|N_{L/K}(\cdot)|_K}$ with $|\cdot|_K$ non-arch., discrete so $|\cdot|_L$ non-arch, discrete.)

and the residue field is finite degree $/\mathbb{F}_p$ since $L/K$ is of finite degree.

( b/c residue classes in residue field linearly indep $\Longleftrightarrow$ lifts in $L$ are linearly indep $/K$ )

i.e. dependence over $L$ gives non-trivial dependence over $L/\mathfrak{p}_L$ )

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Given any local field, $K$, with discrete val $v$, residue field of char $p$.

Do $\mathrm{char}(K)=0$ argument (show result is finite extn of $\mathbb{Q}_p$) and leave function field case to you.

If $\mathrm{char}(K)=0$, then $\mathbb{Q} \hookrightarrow K$ and restriction of $v$ to $\mathbb{Q}$ is $p$-adic valuation (with $p = \mathrm{char}(k)$) since $p \equiv 0\ (\mathfrak{p})$. Since $K$ complete w.r.t. $v$ then $\mathbb{Q}_p \subseteq K$.

How to show $K/\mathbb{Q}_p$ finite? Use local compactness of $K$ as v.s. $/\mathbb{Q}_p$. (needs proof. "topological linear algebra")

Or alternately by studying extensions of "Henselian fields" where we characterize degree of extension using finite data analogous to ramification / inertia degree in study of ideals.

Now we explore structure of mult. gp $K^\times \subseteq K$, a local field.

Knew from previous investigations that $\mathcal{O}_v$ is D.V.R. so elts in $K^\times$ have form $\pi^n \cdot \varepsilon$ with $n \in \mathbb{Z}$, $\varepsilon \in \mathcal{O}_v^\times$. Don't know much about $\mathcal{O}_v^\times$.

We did see, as example of Hensel's lemma, that if residue field of $K$ is $\mathbb{F}_q$, then $K$ contains $(q-1)^{th}$ roots of unity

$q = p^r$ some prime $p$

(since $X^{q-1} - 1$ splits in residue field, it splits in $K$)

① These are, of course, in $\mathcal{O}_v^\times$

② These $(q-1)^{th}$ rts. of unity give reps. for mult. gp. of residue field so under canonical homom.

$$\begin{aligned} \mathcal{O}_v^\times &\longrightarrow (\mathcal{O}_v/\mathfrak{p}_v)^\times \\ a &\longmapsto a \bmod \mathfrak{p}_v \end{aligned}$$

with kernel $1+\mathfrak{p}_v$, we obtain

$$\mathcal{O}_v^\times = (1+\mathfrak{p}_v) \times \underbrace{\mu_{q-1}}_{(q-1)^{th} \text{ roots of } 1}$$

$(1+\mathfrak{p}_v)$ called $\mathcal{U}^{(1)}$, with $\mathcal{U}^{(n)} = (1+\mathfrak{p}_v^n)$ base of nbhds of $1$ in $\mathcal{O}_v^\times$

For global fields, found gp of units was free $\mathbb{Z}$-module of rank $r+s-1$ with roots of unity in field

so $\mathcal{O}_K^\times \simeq \mu(K) \times \mathbb{Z}^{r+s-1}$

Naive guess for $\mathcal{O}_v^\times \simeq \mu(\mathcal{O}_v) \times \mathbb{Z}_p^{*}$ $\leftarrow$ something, where $p$: char. of residue field.

This is true, but haven't yet determined $\mu(\mathcal{O}_v)$.

Only know it contains $\mu_{q-1}$.

And we don't know what the value of $(*)$ should be.

Want to understand $\mathcal{U}^{(1)} = 1+\mathfrak{p}_v$ as <u>topological</u> group.

<u>Claim</u>: $\mathcal{U}^{(1)}$ is a $\mathbb{Z}_p$-module where $\mathrm{char}(\mathcal{O}_v/\mathfrak{p}_v) = p$.

<u>pf</u>: $\mathcal{U}^{(i)}/\mathcal{U}^{(i+1)} \simeq \mathcal{O}_v/\mathfrak{p}_v$, since with $(\pi) = \mathfrak{p}_v$,

the map $1 + \pi^i \cdot \alpha \longmapsto \alpha \bmod \mathfrak{p}_v$ maps $\mathcal{U}^{(i)}$ to $\mathcal{O}_v/\mathfrak{p}_v$ with kernel $\mathcal{U}^{(i+1)}$.

If $\mathcal{O}_v/\mathfrak{p}_v \simeq \mathbb{F}_q$, then $\mathcal{U}^{(1)}/\mathcal{U}^{(n+1)}$ has order $q^n$, and

is thus a $\mathbb{Z}/q^n\mathbb{Z}$ module: $a : (1+x) \bmod \mathcal{U}^{(n+1)} \longmapsto (1+x)^a \bmod \mathcal{U}^{(n+1)}$

for $a \in \underbrace{\mathbb{Z}/q^n\mathbb{Z}}$ viewed as subset of $\mathbb{Z}$

But $\mathcal{U}^{(1)} = \varprojlim_n \mathcal{U}^{(1)}/\mathcal{U}^{(n+1)}$

and $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/q^n\mathbb{Z}$ $q = p^r$.

and the actions of $\mathbb{Z}/q^n\mathbb{Z}$ are compatible with projection maps forming $\varprojlim_n$

so we get $\mathbb{Z}_p$-action on $\mathcal{U}^{(1)}$, extending natural $\mathbb{Z}$-module structure on $\mathcal{U}^{(1)}$, continue to write $a \in \mathbb{Z}_p : (1+x) \mapsto (1+x)^a$

Want to further argue this is continuous action of topological gps:

Indeed, if $a \equiv a' \bmod q^n \cdot \mathbb{Z}_p$ then $(1+x)^a \equiv (1+x)^{a'} \bmod \mathcal{U}^{(n+1)}$

i.e. nbhd $a + q^n\mathbb{Z}_p$ is mapped onto $(1+x)^a \cdot \mathcal{U}^{(n+1)}$

<u>Theorem</u>: Let $K$ be local function field, so $K \simeq \mathbb{F}_q((t))$ with $q = p^r$

then $K^\times \simeq \mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} \oplus \mathbb{Z}_p^{\mathbb{N}}$ ← countable copies

as topological isomorphism of $\mathbb{Z}_p$ modules.

Remember $\mathbb{F}_q((t))$ : Laurent series in $t$ (as opposed to $\mathbb{F}_q(t)$ – global function field – write rational functions in $t$.)

pf (sketch):

We construct continuous homoms.

$$g_n : \mathbb{Z}_p^{\oplus r} \longrightarrow \mathcal{U}^{(n)} = 1 + t^n \mathbb{F}_q[[t]]$$ for each $n$ s.t. $\gcd(n,p)=1$

by choosing basis $\omega_1, \dots, \omega_r$ for $\mathbb{F}_q/\mathbb{F}_p$

$$(a_1, \dots, a_r) \longmapsto \prod_{i=1}^{r} (1 + \omega_i t^n)^{a_i}$$

Then map $g := \prod_{(n,p)=1} g_n : \underbrace{\prod_{(n,p)=1} \mathbb{Z}_p^r}_{\cong\, \mathbb{Z}_p^{\mathbb{N}},\ \text{compact}} \longrightarrow \mathcal{U}^{(1)}$ $g$: continuous.

show $g$ is isomorphism.

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Thm: If $K$ is $p$-adic local field, $K^\times \simeq \mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} \oplus \underbrace{\mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}_p^d}_{\mathcal{U}^{(1)}}$

if $\deg K/\mathbb{Q}_p = d$, $a \geq 0$.

pf: Key step is assertion that there exists continuous isomorphism

$$\log : \mathcal{U}^{(n)} \longrightarrow \mathfrak{p}_v^n = \pi^n \cdot \mathcal{O}_v \simeq \mathcal{O}_v$$

for $n$ sufficiently large.

If we know this, then $\mathcal{O}_v$ : int. closure of $\mathbb{Z}_p$ in $K$ (proved ~~last time~~ last. since $K$ extends val. on $\mathbb{Q}_p$)

so has integral basis over $\mathbb{Z}_p$. of rank $\deg(K/\mathbb{Q}_p)$ (long ago proof, uses $\mathbb{Z}_p$ is P.I.D. proves $\underline{d} \cdot \mathcal{O}_v \subseteq$ integral basis where $\underline{d}$ is disc. of size $\deg(K/\mathbb{Q}_p)$)

i.e. for $n$ sufficiently large,

$\mathcal{U}^{(n)} \simeq \mathbb{Z}_p^d$. But $\mathcal{U}^{(1)}/\mathcal{U}^{(n)}$ finite, so done by structure of modules over P.I.D.