Extensions of non-arch. local fields.

<u>Neukirch</u>: Don't need completeness, just need Hensel's Lemma to be true. This guaranteed unique ext'n of absolute value v on K to algebraic ext'n L.

<u>Example</u> of field having Hensel's Lemma but not complete:

$$K \subseteq K_v \subseteq \hat{K} \quad \text{where } K_v: \text{separable closure of } K \text{ in } \hat{K}$$

"Henselization of K with respect to v" (compositum of separable ext'ns of K in $\hat{K}$.)

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Either way, start with L/K algebraic ext'n of K (either Henselian or just assume $= \hat{K}$) of deg n.

$$v_L(\alpha) = \frac{1}{n} v_K(N_{L/K}(\alpha))$$

Then as v_L extends v_K, $v_K(K^\times) \subseteq v_L(L^\times)$

Let $e = [v_L(L^\times) : v_K(K^\times)]$ "ramification index"

$f = [\mathcal{O}_L/\mathfrak{p}_L : \mathcal{O}_K/\mathfrak{p}_K]$ "residual degree"

If v_K discrete, so v_L discrete, then recall the valuation takes values in $v_K(\pi_K)\cdot\mathbb{Z}$ and $v_L(\Pi_L)\cdot\mathbb{Z}$ respectively.

so e is equivalently $v_K(\pi_K) / v_L(\Pi_L)$ i.e. $\pi_K = \varepsilon \Pi_L^e$ for some unit $\varepsilon \in \mathcal{O}_L^\times$

i.e. $\mathfrak{p}_K \cdot \mathcal{O}_L = \mathfrak{P}_L^e$

<u>Proposition</u>: $[L:K] \geq ef$, with equality if v is discrete, L/K separable.

proof: $[\underbrace{\mathcal{O}_L/\mathfrak{P}_L}_{=\lambda} : \underbrace{\mathcal{O}_K/\mathfrak{p}_K}_{=\kappa}] = f$. Let $\omega_1, \dots, \omega_f$ be rep's in $\mathcal{O}_L$ for basis for λ/κ

and let $\pi_0, \dots, \pi_{e-1}$ be reps. in $L^\times$ which map to cosets in $v_L(L^\times)/v_K(K^\times)$

Know f is finite a priori, even though κ may not be, but don't yet know e finite.

(Neukirch notes if v discrete, then e finite, take $\pi_0, \dots, \pi_{e-1}$ to be corresp. powers of Π_L: $\Pi_L^0, \dots, \Pi_L^{e-1}$.)

Plan: Show $\omega_j \pi_i$ are linearly independent (so $e \cdot f \le [L:K]$)

Suppose $\exists$ non-trivial linear comb. $\sum_i \sum_j a_{ij} \omega_j \pi_i = 0$ with a_{ij} not all 0. $a_{ij} \in K$

Consider the sums $s_i = \sum_{j=1}^{f} a_{ij} \omega_j$ (not all 0)

Then $v_L(s_i) \in v_K(K^\times)$, since dividing s_i by the a_{ij} with minimal valuation, resulting s_i' is linear comb. of ω_j with coeffs in $\mathcal{O}_K$ (not just K), and one coeff. is 1.

So $s_i' \not\equiv 0$ $\mathfrak{P}$ $\Rightarrow$ s_i' is unit $\Rightarrow$ $v_L(s_i) = v_L(a_{ij}$ with min. val.$)$ $\in v_K(K^\times)$ since $a_{ij} \in K^\times$ and v_L extends v_K.

Now consider $\sum_i s_i \pi_i = 0$.

Must have two summands of equal valuation

(remember if $v_L(x) \neq v_L(y)$ then $v_L(x+y) = \min\{v_L(x), v_L(y)\}$) not inequality!

So suppose $v_L(s_i \pi_i) = v_L(s_j \pi_j)$ some pair of indices i, j.

then $v_L(\pi_i) = v_L(\pi_j) + \underbrace{v_L(s_j) - v_L(s_i)}_{\text{in } v_K(K^\times)} \equiv v_L(\pi_j) \mod v_K(K^\times)$

↯ since π_i, π_j chosen to be distinct. Hence no such nontrivial linear comb. exists

and $\pi_i \omega_j$ are linearly indep.

Now if v_K (and hence v_L) discrete, consider $\mathcal{O}_K$-module

$$M = \sum_i \sum_j \mathcal{O}_K \cdot \omega_j \pi_i \qquad \text{with } \pi_i = \Pi^i, \ (\Pi) = \mathfrak{P}_L.$$

Done if we can show $M = \mathcal{O}_L$. (i.e. $\omega_j \pi_i$ are integral basis)

Play similar game as above, writing $N = \sum_j \mathcal{O}_K \omega_j$

so $$M = N + \Pi \cdot N + \cdots + \Pi^{e-1} N \qquad (*)$$

Now $\mathcal{O}_L = N + \Pi \cdot \mathcal{O}_L$ because ω_j are basis for $\mathcal{O}_L / \Pi \cdot \mathcal{O}_L$ as $\mathcal{O}_K / \pi \mathcal{O}_K$-vector space

so we have recursively substituting:

$$\mathcal{O}_L = N + \Pi(N + \Pi \mathcal{O}_L) = \cdots$$
$$= N + \Pi N + \cdots + \Pi^{e-1} N + \Pi^e \mathcal{O}_L \qquad (**)$$

i.e. comparing $(*)$ and $(**)$

$$\mathcal{O}_L = M + \mathfrak{P}_L^e = M + \mathfrak{p}_K \cdot \mathcal{O}_L.$$

To finish, use Nakayama's Lemma + fact that $\mathcal{O}_L$ is f.gen. $\mathcal{O}_K$-module ~~...~~ as L/K separable.

view $\mathcal{O}_L, M$ as $\mathcal{O}_K$-modules with maximal ideal of $\mathcal{O}_K = \mathfrak{p}_K$.

$$\Rightarrow \mathcal{O}_L = M$$

Worth returning to original pf. for # fields using Chinese Remainder Thm to compare with above!