

It is unique because, if  $K', K''$  are unram. extensions of  $K$  both with residue field  $\lambda$ , then the compositum  $K' \cdot K''$  also has residue field  $\lambda$  and is unramified by earlier result.

Hence  $[K' \cdot K'' : K] = [\lambda : k_K] = [K' : K]$  so  $K' = K''$ .

Example:  $K$ : local field of char. 0 (finite ext'n of  $\mathbb{Q}_p$ )

then residue field has order  $q = p^r$ , some  $r$ . Finite extensions of  $k_K$

of  $\mathbb{F}_q$  ~~extension~~ of deg  $n$  are the splitting fields of  $X^{q^n} - X$  over  $\mathbb{F}_q$ .

(Galois ext'ns with cyclic Galois gp. generated by  $x \mapsto x^q$  "Frobenius elt.")

So by the above correspondence, unramified ext'ns of  $K$  are likewise splitting fields of  $X^{q^n} - X$  with cyclic Galois

gp of order  $n$  and canonical generator  $\sigma$  "Frobenius"

with  $\sigma(\alpha) = \alpha^q \pmod{\mathfrak{f}_n}$   $\forall \alpha \in \mathcal{O}_{K_n}$

where  $\mathfrak{f}_n$ : maximal ideal in  $K_n/K$

Can go further  $\rightarrow$  "tamely ramified extensions"

$L/K$  with  $\lambda_L/k_K$  separable,  $T$ : max. unram. ext'n of  $K$  in  $L$  (by above  $\leftrightarrow$   $k_S$ : sep. closure of  $k_K$  in  $\lambda_L$ )

Ask that  $[L:T]$  is relatively prime to  $p = \text{char}(k_K)$

prove similarly that compositum of tamely ram is tamely ramified.

Just as with global fields, intermediate extensions:

$$K \subseteq T : \text{max. unram in } L \subseteq V : \text{max. tamely ram. in } L \subseteq L$$

"wild ramification"

$$v_K \subseteq \lambda_S : \text{max. sep ext'n} \subseteq \lambda_S \subseteq \lambda$$

$$v_K(K^x) = v_T(T^x) \subseteq v_L(V^x) \subseteq v_L(L^x)$$

"  $v_L(L^x)^{(p)}$  "

$$= \left\{ \alpha \in v_L(L^x) \mid m \cdot \alpha \in v_K(K^x) \right\}$$

$(m/p) = 1$   
 $m \in \mathbb{N}$

Return to our earlier example:

$$\mathbb{Q}_p(\xi) / \mathbb{Q}_p \quad \xi : n^{\text{th}} \text{ root of unity.}$$

Two extreme cases.  $(n/p) = 1$ , then  $\exists$  smallest  $f$  s.t.  $n \mid p^f - 1$   
 (i.e. order of  $p \pmod n$ )

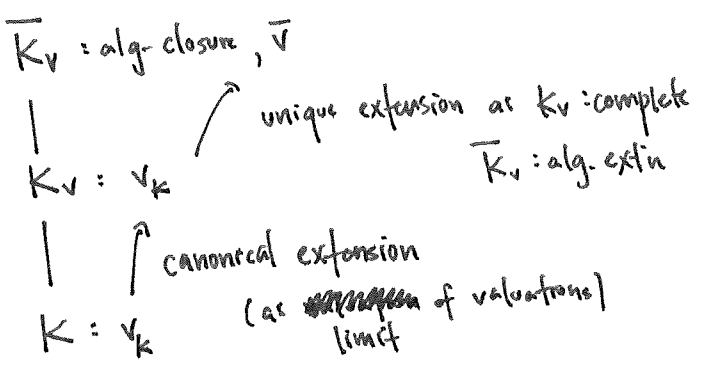
then  $\mathbb{Q}_p(\xi)$  is unramified extension  
 with degree  $f$  over  $\mathbb{Q}_p$ .

if  $n = p^m$ , then  $\mathbb{Q}_p(\xi) / \mathbb{Q}_p$  is "totally ramified" - no proper unramified extns of  $\mathbb{Q}_p$ .

of degree  $\varphi(p^m) = p^{m-1} \cdot (p-1)$

Now let  $K$  be field with valuation  $v$  (or  $|\cdot|$  if Archimedean)

and let  $K_v$  denote its completion w.r.t.  $v$ .



If  $L/K$  algebraic extension,

let  $\tau: L \rightarrow \bar{K}_v$  be a  $K$ -embedding

Then we may attach valuation

to  $L$  by  $v_L = \bar{v} \circ \tau$

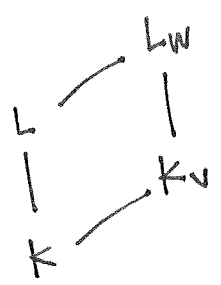
(and valuation extends  $v_K$  since  $\tau$  is a  $K$ -embedding)

This is very reminiscent of early proofs using geometry of numbers, but not using only  $\mathbb{R}, \mathbb{C}$  embeddings, but for arbitrary complete local ring.

For finite extensions  $L/K$ ,

then let  $L_{v_L}$ : completion of  $L$  w.r.t.  $v_L$

So we have the extensions of fields setting  $w := v_L$ .



and valuation on  $L_w$  extends that on  $K_v$  so must be unique ext'n in  $L_w/K_v$ .

then  $\tau$  extends to continuous map

$$\tau: L_{v_L} \rightarrow \bar{K}_v$$

$$x = \lim_{n \rightarrow \infty} x_n \mapsto \tau(x) = \lim_{n \rightarrow \infty} \tau(x_n)$$

In particular if  $L_w/K_v$  finite extension, then  $\text{deg } n$

$$|x|_w = \sqrt[n]{|N_{L_w/K_v}(x)|_v}$$

Also  $L_w = L \cdot K_v$  since  $L/K$  finite

$\Rightarrow L \cdot K_v$  complete and contains  $L$ , and  $\subseteq L_w$ .  
part of unique ext'n theorem.

if  $L/K$  infinite take  $L_{v_L}$  to be union of all finite subextensions' completions.

Just as we wanted to identify conjugate embeddings in  $\mathbb{C}$ , want to

do the same in  $\bar{K}_v$ . What are conjugate embeddings? conjugation is just non-trivial elt. of  $\text{Gal}(\mathbb{C}/\mathbb{R})$

Now take  $\sigma \in \text{Gal}(\bar{K}_v/K_v)$ .

Say  $\tau, \tau'$ : two  $K$ -embeddings of  $L/K$  are "conjugate" if  $\exists \sigma \in \text{Gal}(\bar{K}_v/K_v)$  such that  $\tau' = \sigma \circ \tau$ .

Classification theorem (version 1)  $L/K$  algebraic.  $v$ : valuation of  $K$ ,

Every ext'n of  $v$  to  $L$  is given by  $\bar{v} \circ \tau$ ,  $\tau: K$ -embedding of  $L \rightarrow \bar{K}_v$

and two valuations  $\bar{v} \circ \tau, \bar{v} \circ \tau'$  are equal iff  $\tau, \tau'$  are conjugate.

classification theorem (version 2) ~~When~~  $L = K(\alpha)$  (e.g.  $L/K$  separable)

with  $f(x) \in K[x]$  the minimal poly. of  $\alpha$ .

with  $f(x) = f_1(x)^{m_1} \dots f_r(x)^{m_r}$  irreducible factors  $f_i$  in  $K_v[x]$ .  
(~~all~~ ~~if~~ ~~separable~~ if  $f$  separable)  
 $m_i = 1$

then valuations of  $L$  extending  $v$  are in 1-1 correspondence with irreducible factors  $f_i$ .

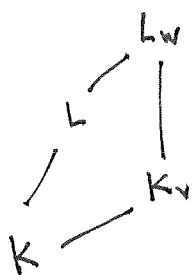
pf: (version 1)  $\rightarrow$  (version 2):  $K$ -embeddings  $\tau: L \rightarrow \bar{K}_v$  are given by zeros  $\beta$  of  $f(x)$  lying in  $\bar{K}_v$ : so  $K$ -embedding def'd by sending  $\alpha \mapsto \beta$  for each  $\alpha$ . Two embeddings are conjugate precisely when the zeros  $\beta, \beta'$  are from the same irreducible factor  $f_i$ .

proof of classification 1: <sup>(1)</sup> Every extension  $w = \bar{v} \circ \tau$ ,  $\tau: K$ -embedding:

Assume  $L/K$  finite. If  $L/K$  infinite, proof is same, but meaning of  $L_w$  changes.  
 $w$ : valuation on  $L$ ,  $v$ : val. on  $K$ .

Then any  $K_v$ -embedding  $\tau: L_w \rightarrow \bar{K}_v$  (which exists since  $L_w = K_v \cdot L$ ,  $L/K$  alg.)  
 gives valuation  $\bar{v} \circ \tau$  extending  $v$  on  $K_v$ , so must equal

$w$  = the unique ext'n of  $K_v$ -valuation  $v$  to  $L_w$ .



If we restrict  $w$  to  $L$ , then  $\tau$  restricts to  $K$ -embedding  $\tau: L \rightarrow \bar{K}_v$  with  $w = \bar{v} \circ \tau$ .

(2) ~~embed~~ equal iff  $\tau, \tau'$  conjugate.  
 $\bar{v} \circ \tau,$   
 $\bar{v} \circ \tau'$

( $\Leftarrow$ ) Given  $\tau$ : embedding,  $\sigma \in \text{Gal}(\bar{K}_v/K_v)$ , consider whether  $\tau, \sigma \circ \tau$  are equal.

$\bar{v}$  is unique extension from  $K_v$  to  $\bar{K}_v$ ,  
 of  $v$

so for any  $\sigma \in \text{Gal}(\bar{K}_v/K_v)$ ,  $\bar{v} = \bar{v} \circ \sigma \Rightarrow \bar{v} \circ \tau = \bar{v} \circ \sigma \circ \tau$   
 i.e. valuations are equal.

( $\Rightarrow$ ) Given  $\tau, \tau': L \rightarrow \bar{K}_v$  s.t.  $\bar{v} \circ \tau = \bar{v} \circ \tau'$

idea: produce  $K$ -isomorphism of  $\tau L \rightarrow \tau' L$ , extend to autom. of  $\bar{K}_v, \sigma$ , over  $K$  with desired property.

Take  $\sigma = \tau' \circ \tau^{-1}$ . Then want to extend to

map  $\sigma: \tau L \cdot K_v \rightarrow \tau' L \cdot K_v$

since  $\tau L$  is dense in  $\tau L \cdot K_v$  as  $\tau L$  contains  $K$ .

thus every elt  $x \in \tau L \cdot K_v$  is expressible as limit of elts in  $\tau L$ ,

i.e.  $x = \lim_{n \rightarrow \infty} \tau x_n, x_n \in L \forall n$

Since  $\bar{v} \circ \tau = \bar{v} \circ \tau'$  then  $\{\tau' x_n\} = \{\sigma \tau x_n\}$  converges in  $\bar{v} \circ \tau'$  topology

Call resulting limit  $\sigma x$ . of  $\tau' L \cdot K_v$

Then  $x \mapsto \sigma x$  is our desired isomorphism  $\tau L \cdot K_v \rightarrow \tau' L \cdot K_v$ .

and it leaves  $K_v$  fixed. Now extend to autom.  $\tilde{\sigma}$  of  $\text{Gal}(\bar{K}_v/K_v)$ .

Fancy algebraic formulation: tensor products of vector spaces -

Have natural homomorphism

$$L \otimes_K K_v \longleftrightarrow L_w = L \cdot K_v$$

$$a \otimes b \longmapsto ab \quad (\text{or maybe better } \tau(a) \otimes b)$$

to emphasize  $K$ -embedding

$L$  as  $K$ -vector space now viewed as  $K_v$ -vector space (extension of scalars)

If we do this for all places  $w$  over  $v$  for  $L$ ,

then have map

$$\phi: L \otimes_K K_v \longrightarrow \prod_{w|v} L_w$$

Proposition: If  $L/K$  separable, then  $\phi$  is isomorphism.

Why is such an extension guaranteed? One alternative is to say equivalent/conjugate extensions require isom. of compositum  $\tau L \cdot K_v \rightarrow \tau' L \cdot K_v$  in fixed alg. closure.

Then  $\tau' L \cdot K_v / K_v$  is finite extn. so has unique abs. value extending one on  $K_v$ .

(As in Jacobson, B.A.-I p. 585)