Just as in the treatment via ideals, if L/K Galois then can make more precise conclusions about extensions by studying action on valuations.

Always two cases — L/K finite — familiar properties of Galois ext'ns 1-1 corresp. of intermediate fields and subgps. of $Gal(L/K)$

L/K infinite — this 1-1 corresp. is false w/o modification.

Introduce Krull topology on $Gal(L/K)$, where base of open nbhds of σ consists of cosets $\sigma \cdot Gal(L/M)$ with M/K: finite, Galois subext. in L/K.

Then $Gal(L/K)$ is compact, Hausdorff in this topology and 1-1 corresp. is restored by considering only closed subgroups w.r.t. Krull topology.

("closed" is important condition in category of topological gps. E.g. $H \triangleleft G$, H not closed then G/H won't be Hausdorff in quotient topology)

Proposition: G acts transitively on the extensions $w|v$.

First note that $\sigma \in G := Gal(L/K)$ acts by $|\alpha|_{\sigma \circ w} := |\sigma^{-1}(\alpha)|_w$

so $|\alpha|_{\sigma(\tau(w))} = |\sigma^{-1}(\alpha)|_{\tau(w)} = |\underbrace{\tau^{-1}\sigma^{-1}}_{(\sigma\tau)^{-1}}(\alpha)|_w = |\alpha|_{(\sigma\tau)(w)}$

Or: act on right: $|\alpha|_{w \circ \sigma} := |(\alpha)\sigma|_w$ (Neukirch)

Either way, clearly $w \circ \sigma$ extends v since σ fixes K.

pf. of Proposition: In case of ideals, showed $Gal(L/K)$ acts transitively on $\mathfrak{P}_i$ in $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$ by using fact that any two $\mathfrak{P}_i, \mathfrak{P}_j$ have $\mathfrak{P}_i \cap \mathcal{O}_K = \mathfrak{P}_j \cap \mathcal{O}_K = \mathfrak{p}$. Now use Chinese Remainder Thm to arrange x s.t. $x \equiv 0 \ \mathfrak{P}_i$ $x \equiv 1 \ \sigma(\mathfrak{P}_j) \ \forall \sigma \in Gal(L/K)$ if $\mathfrak{P}_i$ not among $\sigma(\mathfrak{P}_j)$

Then taking norms, get $N_{L/K}(x)$ both in $\mathfrak{p}$, not in $\mathfrak{p}$. ↯

$$N_{L/K}(x) = \prod_{\sigma \in Gal(L/K)} \sigma(x)$$

"in $\mathfrak{p}$": using $x \equiv 0\ \mathfrak{P}_i$; "not in $\mathfrak{p}$": using $x \equiv 1 \bmod \sigma(\mathfrak{P}_j)$

We do the same in case L/K finite, but recall that our replacement for CRT is:

Given inequivalent valuations on L, $\epsilon > 0$, $a_1, \dots, a_n \in L$ we can find x s.t. $|x - a_i|_i < \epsilon \quad \forall\ i = 1, \dots, n$.

Indeed if w, w' not conjugate then $\{ w \circ \sigma \mid \sigma \in Gal(L/K) \}$ is completely disjoint from $\{ w' \circ \sigma \mid \sigma \in Gal(L/K) \}$, so by approximation thm., $\exists\ x \in L$ with $|\sigma x|_w < 1$, $|\sigma x|_{w'} > 1 \quad \forall\ \sigma \in Gal(L/K)$.

Taking norms, let $\alpha = N_{L/K}(x) = \prod_{\sigma \in Gal(L/K)} \sigma(x)$. Then $|\alpha_v| = \prod_{\sigma \in Gal(L/K)} |\sigma x|_w^n < 1$ and $\prod_{\sigma \in Gal(L/K)} |\sigma x|_{w'}^n > 1$. ↯

In infinite case, use above result + little topology: if L/K infinite, w, w' : vals. on L

Let M/K : finite Galois subextn. $X_M = \{ \sigma \in G \mid w \circ \sigma|_M = w'|_M \}$

Know X_M non-empty by above, and in fact closed since if $\gamma \in G \setminus X_M$ then open set $\gamma \cdot Gal(L/M)$ is in $G \setminus X_M$, and so the complement of X_M is the union of open sets.

__Note:__ G acts trans. if $\bigcap X_M \neq \phi$

Now if $\bigcap_M X_M = \phi$ then since G compact, X_M closed $\forall\ M$,

$\Rightarrow\ \exists$ finite intersection $\bigcap_{i=1}^{r} X_{M_i} = \phi$. ↯ since $\bigcap_{i=1}^{r} X_{M_i} = X_{M_1 \cdot M_2 \cdots M_r}$

the finite intersection claim is making use of Heine-Borel property.

Now we may define decomposition gp. assoc. to valuation w extending v:

$$G_w = \{ \sigma \in \mathrm{Gal}(L/K) \mid w \circ \sigma = w \}$$

If w, v non-archimedean (so have valuation, not just abs. value, with ring of intgrs., etc.)

$$I_w = \{ \sigma \in G_w \mid \sigma x \equiv x \mod \mathfrak{P}_L \ \forall\, x \in \underset{\sim}{\mathcal{O}_L} \}$$

val. ring of L w.r.t. w

"inertia gp" —: kernel of G_w under canon. homom.

$$R_w = \{ \sigma \in G_w \mid \frac{\sigma x}{x} \equiv 1 \mod \mathfrak{P}_L \ \forall\ x \in L^\times \}$$

"ramification gp"

with $w \circ \sigma = w$ implying that σ fixes $\mathcal{O}_L$ in particular, so I_w is well-defined and that $\frac{\sigma \cdot x}{x} \in \mathcal{O}_L \ \forall\ x \in L^\times$ so R_w well-defined.

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Further show G_w, I_w, R_w are closed in Krull topology and well-behaved with respect to commutative diagrams: L/K, L'/K' Galois ext'ns such that, for some homom. τ, the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\tau} & L' \\ | & & | \\ K & \xrightarrow{\tau} & K' \end{array}$$

(on L, extending one on K)

then $\exists$ homoms $\tau^* : \mathrm{Gal}(L'/K') \to \mathrm{Gal}(L/K)$

$$\sigma' \mapsto \tau^{-1} \sigma' \tau$$

which restrict to homoms on G_w', I_w', R_w'.

If τ isomorphism, then all induced maps are isoms.

Special case: τ: itself Galois autom. $\quad G_{w \circ \tau} = \tau^{-1} G_w \tau$ etc.

Can also apply diagram to case of tower with completions:

$$\begin{tikzcd}[row sep=small] & L_w \arrow[dd, dash] \\ L \arrow[ur, dash] \arrow[dd, dash] & \\ & K_v \\ k \arrow[ur, dash] & \end{tikzcd}$$

then get map $\quad \mathrm{Gal}(L_w/K_v) \longrightarrow \mathrm{Gal}(L/k)$

$$\sigma \longmapsto \sigma|_L$$

since homom. $\tau : L \to L_w$ is inclusion,

so τ^{-1} : restriction.

<u>Proposition</u>:

$$G_w(L/k) \simeq G_w(L_w/K_v)$$
$$I_w(L/k) \simeq I_w(L_w/K_v)$$
$$R_w(L/k) \simeq R_w(L_w/K_v)$$

<u>pf</u>: Key fact is that $\sigma \in G_w(L/k) \iff \sigma$ continuous w.r.t. w. Immediate that $\sigma \in G_w(L/k)$ is continuous w.r.t w.

If $\sigma \in \mathrm{Gal}(L/k)$, continuous, then

$$|x|_w < 1 \Rightarrow \{x^n\} \to 0 \text{ in } w\text{-top.}$$
$$\Rightarrow \{\sigma x^n\} \to 0 \text{ in } w\text{-top}$$
$$\Rightarrow |\sigma x|_w < 1 \quad \text{i.e.} \quad |x|_{w\circ\sigma} < 1$$

$\Rightarrow \quad w \sim w\circ\sigma \quad$ (and in fact equal since they agree on k)

i.e. $\sigma \in G_w(L/k)$.

But now L dense in L_w, so each $\sigma \in G_w(L/k)$ admits unique extension to continuous $G_w(L_w/K_v)$ (also preserves I_w, R_w)

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Big picture for non-arch. valuations :

same residue field.

$$\begin{tikzcd}
\mathfrak{P} & L \arrow[r, dash] \arrow[d, dash, "e"'] & L_{\mathfrak{P}} \arrow[d, dash, "e"] & \\
\mathfrak{P}_I & L^{I_{\mathfrak{P}}(L_{\mathfrak{P}}/K_{\mathfrak{p}})} \arrow[r, dash] \arrow[d, dash, "f"'] & L_{\mathfrak{P}}^{I_{\mathfrak{P}}(L_{\mathfrak{P}}/K_{\mathfrak{p}})} \arrow[r, dash] \arrow[d, dash, "f"] & \lambda_L \arrow[d, dash, "f"] \\
\mathfrak{P}_D & L^{G_{\mathfrak{P}}(L_{\mathfrak{P}}/K_{\mathfrak{p}})} \arrow[r, dash] \arrow[d, dash, "g"'] & K_{\mathfrak{p}} \arrow[r, dash] & \kappa_K \\
\mathfrak{p} & K \arrow[ur, dash] & &
\end{tikzcd}$$

$\left.\right\}$ $Gal \cong G_{\mathfrak{P}}/I_{\mathfrak{P}}$