

Just as in the treatment via ideals, if  $L/K$  Galois then can make more precise conclusions about extensions by studying action on valuations.

Always two cases -  $L/K$  finite - familiar properties of Galois extns  
1-1 corresp. of intermediate fields and subgps. of  $Gal(L/K)$

$L/K$  infinite - this 1-1 corresp. is false w/o modification.

Introduce Krull topology on  $Gal(L/K)$ , where base of open nbhds of  $\sigma$  consists of cosets  $\sigma \cdot Gal(L/M)$  with  $M/K$  : finite, Galois subext. in  $L/K$ .

Then  $Gal(L/K)$  is compact, Hausdorff in this topology and 1-1 corresp. is restored by considering only

closed subgroups w.r.t. Krull topology. ("closed" is important condition in category of topological gps. E.g.  $H \triangleleft G$ ,  $H$  not closed then  $G/H$  won't be Hausdorff in quotient topology)

Proposition :  $G$  acts transitively on the extensions  $w/v$ .

First note that  $\sigma \in G := Gal(L/K)$  acts by  $|\alpha|_{\sigma \circ w} := |\sigma^{-1}(\alpha)|_w$

$$\text{so } |\alpha|_{\sigma(\tau(w))} = |\sigma^{-1}(\alpha)|_{\tau(w)} = |\underbrace{\tau^{-1} \sigma^{-1}(\alpha)}_{(\sigma\tau)^{-1}}|_w = |\alpha|_{(\sigma\tau)(w)}$$

Or : act on right :  $|\alpha|_{w \circ \sigma} := |(\alpha)\sigma|_w$  (Neukirch)

Either way, clearly  $w \circ \sigma$  extends  $v$  since  $\sigma$  fixes  $K$ .

pf. of Proposition : In case of ideals, showed  $Gal(L/K)$  acts transitively

on  $\mathfrak{p}_i$  in  $\mathfrak{p} \mathcal{O}_L = \mathfrak{p}_1^{a_1} \dots \mathfrak{p}_r^{a_r}$  by using fact that any two  $\mathfrak{p}_i, \mathfrak{p}_j$

have  $\mathfrak{p}_i \cap \mathcal{O}_K = \mathfrak{p}_j \cap \mathcal{O}_K = \mathfrak{p}$ . Now use Chinese Remainder Thm to arrange  $x$  s.t.  $x \equiv 0 \pmod{\mathfrak{p}_i}$   $x \equiv 1 \pmod{\mathfrak{p}_j}$   $\forall \mathfrak{p}_j \neq \mathfrak{p}_i$  if  $\mathfrak{p}_i$  not among  $\mathfrak{p}_j$

Then taking norms, get  ~~$N_{L/K}(x)$~~   $N_{L/K}(x)$  both in  $\mathbb{F}$ , not in  $\mathbb{F}$ .  $\downarrow$   
 $\prod_{\sigma \in \text{Gal}(L/K)} \sigma(x)$  using  $x \equiv 0 \pmod{\mathfrak{p}_i}$  using  $x \equiv 1 \pmod{\mathfrak{p}_j}$

We do the same in case  $L/K$  finite, but recall that our replacement for CRT

is: Given inequivalent valuations on  $L$ ,  $\epsilon > 0$ ,  ~~$a_1, \dots, a_n \in L$~~   $a_1, \dots, a_n \in L$   
 we can find  $x$  s.t.  $|x - a_i|_i < \epsilon \quad \forall i = 1, \dots, n$ .

Indeed if  $w, w'$  not conjugate then  $\{w \circ \sigma \mid \sigma \in \text{Gal}(L/K)\}$  is completely disjoint  
 from  $\{w' \circ \sigma \mid \sigma \in \text{Gal}(L/K)\}$ , so by approximation thm.,  $\exists x \in L$   
 with  $| \sigma x |_w < 1, | \sigma x |_{w'} > 1 \quad \forall \sigma \in \text{Gal}(L/K)$ .

Taking norms, let  $\alpha = N_{L/K}(x)$ . Then  $|\alpha_v| = \prod_{\sigma \in \text{Gal}(L/K)} | \sigma x |_w < 1$   
 $\prod_{\sigma \in \text{Gal}(L/K)} \sigma(x)$  and  $\prod_{\sigma \in \text{Gal}(L/K)} | \sigma x |_{w'} > 1$ .  $\downarrow$

In infinite case, use above result + little topology: if  $L/K$  infinite,  $w, w'$ : vals. on  $L$

Let  $M/K$ : finite Galois subext'n.  $X_M = \{ \sigma \in G \mid w \circ \sigma|_M = w'|_M \}$

Know  $X_M$  non-empty by above, and in fact closed Note:  $G$  acts trans. if  $\bigcap X_M \neq \emptyset$   
 since if  $\sigma \in G \setminus X_M$  then open set  $\sigma \cdot \text{Gal}(L/M)$  is in

$G \setminus X_M$ , and so the complement of  $X_M$  is the union of open sets.

Now if  $\bigcap_M X_M = \emptyset$  then since  $G$  compact,  $X_M$  closed  $\forall M$ ,

$\Rightarrow \exists$  finite intersection  $\bigcap_{i=1}^r X_{M_i} = \emptyset$ .  $\downarrow$  since  $\bigcap_{i=1}^r X_{M_i} = X_{M_1 \cdot M_2 \cdots M_r}$

the finite intersection claim is making use of Heine-Borel property.

Now we may define decomposition gp. assoc. to valuation  $w$  extending  $v$ :

$$G_w = \{ \sigma \in \text{Gal}(L/K) \mid w \circ \sigma = w \}$$

If  $w, v$  non-archimedean (so have valuation, not just abs. value, with ring of ints., etc.)

$$I_w = \{ \sigma \in G_w \mid \sigma x \equiv x \pmod{\mathfrak{f}_L} \ \forall \ x \in \mathcal{O}_L \}$$

val. ring of  $L$  w.r.t.  $w$

"inertia gp"  $\rightarrow$  kernel of  $G_w$  under canon. homom.

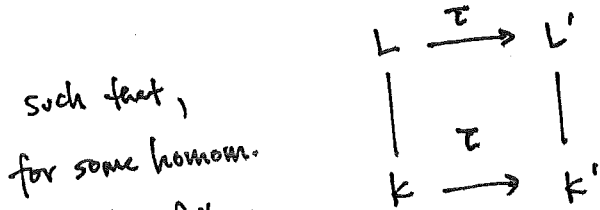
$$R_w = \{ \sigma \in G_w \mid \frac{\sigma x}{x} \equiv 1 \pmod{\mathfrak{f}_L} \ \forall \ x \in L^\times \}$$

"ramification gp"

with  $w \circ \sigma = w$  implying that  $\sigma$  fixes  $\mathcal{O}_L$  in particular, so  $I_w$  is well-defined and that  $\frac{\sigma x}{x} \in \mathcal{O}_L \ \forall \ x \in L^\times$  so  $R_w$  well-defined.

Further show  $G_w, I_w, R_w$  are closed in Krull topology and well-behaved

with respect to commutative diagrams:  $L/K, L'/K'$  Galois extns



then  $\exists$  homoms  $\tau^* : \text{Gal}(L'/K') \rightarrow \text{Gal}(L/K)$   
 $\sigma' \mapsto \tau^{-1} \sigma' \tau$

$\tau$ , the following diagram commutes:  
(on  $L$ , extending one on  $K$ )

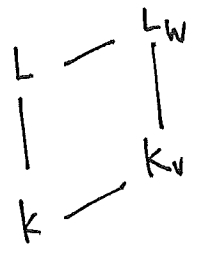
which restrict to homoms on  $G'_w, I'_w, R'_w$ .

If  $\tau$  isomorphism, then all induced maps are isoms.

Special case:  $\tau$ : itself Galois autom.

$$G_{w \circ \tau} = \tau^{-1} G_w \tau \text{ etc.}$$

Can also apply diagram to case of tower with completions:



then get map  $\text{Gal}(L_w/k_v) \rightarrow \text{Gal}(L/k)$

$$\sigma \mapsto \sigma|_L$$

since homom.  $\tau: L \rightarrow L_w$  is inclusion.

so  $\tau^{-1}$ : restriction.

Proposition:

$$G_w(L/k) \cong G_w(L_w/k_v)$$

$$I_w(L/k) \cong I_w(L_w/k_v)$$

$$R_w(L/k) \cong R_w(L_w/k_v)$$

pf: Key fact is that  $\sigma \in G_w(L/k) \iff \sigma|_L$  is continuous w.r.t.  $w$ .

Immediate that  $\sigma \in G_w(L_w/k_v)$  is continuous w.r.t.  $w$ .

If  $\sigma \in \text{Gal}(L/k)$ , continuous, then

$$|x|_w < 1 \Rightarrow \{x^n\} \rightarrow 0 \text{ in } w\text{-top.}$$

$$\Rightarrow \{\sigma x^n\} \rightarrow 0 \text{ in } w\text{-top}$$

$$\Rightarrow |\sigma x|_w < 1 \text{ i.e. } |x|_{w \circ \sigma} < 1$$

$\Rightarrow w \circ \sigma \sim w \circ \sigma$  (and in fact equal since they agree on  $k$ )

i.e.  $\sigma \in G_w(L/k)$ .

But now  $L$  dense in  $L_w$ , so each  $\sigma \in G_w(L/k)$  admits unique extension to continuous  $G_w(L_w/k_v)$  (also preserves  $I_w, R_w$ )

Big picture for non-arch. valuations:

