Thm. 4.4 is corollary to result that, given any measurable set $S$ in Euclidean space, if $\mu(S) > \mathrm{vol}(\Gamma)$ then $\exists\ x, y \in S$ s.t. $x - y \in \Gamma$.

To show Thm 4.4 apply result to $S = \frac{1}{2}X$ (scale all distances by $1/2$)

so that $x - y \in X$ if $x, y \in S$. Note: $\mu(S) = 2^{-n}\mu(X)$.

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Corollary: For every ideal $\mathfrak{a} \neq 0$ in $\mathcal{O}_K$, $\exists\ a \neq 0 \in \mathfrak{a}$ s.t.

$$|N_{K/\mathbb{Q}}(a)| \leq \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|}\, N(\mathfrak{a})$$

pf: choose $c_\tau$ with $c_\tau = c_{\bar{\tau}}$ and $\prod_\tau c_\tau = \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|}\, N(\mathfrak{a}) + \varepsilon$

for any $\varepsilon > 0$. Apply previous result. Since true for all $\varepsilon$, and

$|N_{K/\mathbb{Q}}(a)|$ is positive integer, then we obtain desired inequality $\leq$.

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To finish finiteness of class gp, recall we wanted to show each class in $J_K/K^\times$ contains an integral ideal with abs. norm $\leq$ some fixed const. (depending on $K$, but not on $\mathfrak{a}$)

Given fractional ideal $\mathfrak{a}$, find "denominator" $\gamma$ for its inverse in $J_K$ $\qquad \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|} = C_K$

i.e. $\gamma \in \mathcal{O}_K$ so that $\mathfrak{b} = \gamma \cdot \mathfrak{a}^{-1} \subseteq \mathcal{O}_K$.

Now $\exists\ \beta \in \mathfrak{b}$, $\neq 0$ with $|N_{K/\mathbb{Q}}(\beta)| \leq C_K \cdot N(\mathfrak{b})$ (by corollary)

i.e. $N((\beta)\mathfrak{b}^{-1}) = N \cdot (\beta \cdot \mathfrak{b}^{-1}) \leq C_K$

and $\beta \cdot \mathfrak{b}^{-1} = \beta/\gamma \cdot \mathfrak{a} \in [\mathfrak{a}]$ in $J_K/K^\times$.

To analyze the units $\mathcal{O}_K^\times$ in $K^\times$, need multiplicative version of previous set-up. Now have map (mult. homom.)

$$K^\times \longrightarrow K_{\mathbb{C}}^\times = \prod_\tau \mathbb{C}^\times \longrightarrow \prod_\tau \mathbb{R}$$

$$\alpha \longmapsto \tau(\alpha) \quad ; \quad z \longmapsto \log|z|$$

and we have comm. diagram:

$$\begin{tikzcd} K^\times \arrow[r] \arrow[d, "N_{K/\mathbb{Q}}"'] & \prod_\tau \mathbb{R} \arrow[d, "\mathrm{Tr}"] \\ \mathbb{Q}^\times \arrow[r, "\log|\cdot|"'] & \mathbb{R} \end{tikzcd}$$

Again, we work with $K_{\mathbb{R}}^\times$ so entries at $\tau, \bar\tau$ differ by cx. conjugation.

Since $|z| = |\bar z|$, then upon taking log map, elts in $\tau, \bar\tau$ components are equal, so identify them with $\mathbb{R}$ via $(x,x) \longmapsto 2x$,

$$K_{\mathbb{R}}^\times \to \Big[\prod_\tau \mathbb{R}\Big]^+ \sim \text{fixed by conj.} \qquad \text{and} \qquad \Big[\prod_\tau \mathbb{R}\Big]^+ \cong \mathbb{R}^{r+s}$$

i.e. $K_{\mathbb{R}}^\times \longrightarrow \mathbb{R}^{r+s}$

$$(x_\tau) \longmapsto (\underbrace{\log|x_1|, \dots, \log|x_r|}_{\text{real embeds } r}, \underbrace{\log|x_{r+1}|^2, \dots, \log|x_{r+s}|^2}_{\text{pairs of cx. embeds } s})$$

Then $\mathcal{O}_K^\times = \{ \varepsilon \in \mathcal{O}_K \mid N_{K/\mathbb{Q}}(\varepsilon) = \pm 1 \}$

maps to $S \subseteq K_{\mathbb{R}}^\times := \{ y \in K_{\mathbb{R}}^\times \mid N(y) = \pm 1 \}$

maps to $H = \{ (x_\tau) \in [\prod_\tau \mathbb{R}]^+ \mid \mathrm{Tr}(x_\tau) = 0 \}$
"hyperplane"

We want to show image of $\mathcal{O}_K^\times$ in $H$ is full lattice (so rank $r+s-1$),

call it $\Gamma$, and that $\mathcal{O}_K^\times = \Gamma \times \mu(K)$ ← roots of unity in $K$

Let's first show: The following sequence is exact:

$$1 \to \mu(K) \to \mathcal{O}_K^\times \xrightarrow{\log|\cdot| \circ (\tau)'s} \Gamma \to 0$$

pf: clearly $\mu(K)$ is in kernel, since $|\tau(\xi)| = 1$ for any rt. of unity and any $\tau$ (gp. homomorphism)

And kernel consists of elts $\alpha$ s.t. $|\tau(\alpha)| = 1 \quad \forall \tau$.

This set must be finite since has bounded norm in $K_{\mathbb{R}}$, and each $\alpha \in \mathcal{O}_K^\times \subseteq \mathcal{O}_K$, a (full) lattice in $K_{\mathbb{R}}$. (lattices are discrete)

But only finite subgps of $K$ are contained in $\mu(K)$.

Thm: $\Gamma$: image of $\mathcal{O}_K^\times$ in $\mathbb{R}^{r+s}$, is full lattice in $H \cong \mathbb{R}^{r+s-1}$

pf: To see that $\Gamma$ is lattice, show it is a discrete subgp.

(that these are equivalent is discussed in Prop. 4.2 of Neukirch)

One characterization of discrete is that its intersection with any compact set is finite.

So suffices to consider finiteness w.r.t. $S_c = \{(x_\tau) \mid |x_\tau| \leq c\}$ for any fixed constant $c > 0$

Do this in $K_{\mathbb{R}}^\times$ or in $K_{\mathbb{C}}^\times$, since $K_{\mathbb{R}}^\times$ arises as restriction.

preimage of $S_c$ under log in $K_{\mathbb{C}}^\times$ is $\{(z_\tau) \in \prod_\tau \mathbb{C}^\times \mid e^{-c} \leq |z_\tau| \leq e^{c}\}$

But this set contains only finitely many points in $\phi(\mathcal{O}_K^\times)$ since, again, $\phi(\mathcal{O}_K^\times)$ is subset of $\phi(\mathcal{O}_K)$. Much more painful to show full.