

4.3.3

1) Let  $f(z) = w = z^2 + z$ , and let g(z) = f(z) + 1/4. Clearly, f is bijective on a domain D if and only if g is. However,

 $g(z) = (z + \frac{1}{2})^2$ ,

meaning g maps two poins z and z' to the same point if and only if 1/2 + z = -1/2 - z', that is, z and z' are on opposite sides of -1/2. In particular, this means that the radius r of our circle around the origin must be less than 1/2, since otherwise the domain of g will contain an open neighborhood around -1/2. However, if  $r \leq 1/2$ , then for each point |z| < r,  $\operatorname{Re}(1/2+z)>0$  and  $\operatorname{Re}(-1/2-z)<0$ . Thus, g and therefore f does not repeat any values on |z| < 1/2, so this is the largest disk around the origin for which the given mapping is one

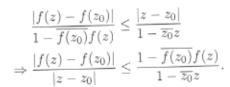
2) Let  $z_1 = x + iy$  and  $z_2 = a + bi$  for  $a, b, x, y \in \mathbb{R}$ . Observe that

$$\begin{split} e^{z_1} &= e^{z_2} \\ \Leftrightarrow e^x(\cos(y) + i\sin(y)) &= e^a(\cos(b) + i\sin(b)). \end{split}$$

Since the magnitudes must be equal, we know x = a. Furthermore, if cos(y) = cos(b) and  $\sin(y) = \sin(b)$ , then necessarily  $y = 2\pi k + b$  for some  $k \in \mathbb{Z}$ . Thus, we will certainly have repeated values if the radius r of the circle around the origin is more than  $\pi$ , however we cannot have repeated values if  $r \leq \pi$ . Thus,  $r = \pi$  is the largest value for which  $w = e^z$  is one to one on the circle  $|z| < \pi$ .

4.3.4

1) Observe that by (36), for any |z| < 1 and  $|z_0| < 1$ , we have





As this holds for arbitrary z and  $z_0$ , we may take the limit as z goes to  $z_0$ , yielding

$$|f'(z)| \le \frac{1 - |f(z)|^2}{1 - |z|^2},$$

which is equivalent to the desired result. Furthermore, if |z| = 1, then by the convention established in this class, if |z| = 1, then 1/(1-|z|) = 1/0 is infinite. In this case, the identity trivially holds.

2)

Define  $F(z):\{z\in\mathbb{C}|\mathrm{Im}(z)>0\}\to\{z\in\mathbb{C}|\mathrm{Im}(z)\geq0\}$  as

$$F(z) = \frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}} \frac{z - \overline{z_0}}{z - z_0}.$$

It is clear from this expression that  $|F(z)| \to 1$  as  $\text{Im}(z) \to 0$  of  $|z| \to \infty$  (where  $z_0$  is fixed). We can rearrange the identity to get

$$\left|\frac{\frac{f(z)-f(z_0)}{z-z_0}}{f(z)-\overline{f(z_0)}}\right| \leq \left|\frac{1}{z-\overline{z_0}}\right|$$

As f is differentiable, F has a removable singularity at  $z_0$ , so by a trivial continuation we can assume F is analytic on the upper half plane. Thus, we have an analytic function satisfying

$$\limsup_{z \to a} |F(z)| \le 1$$

on all points  $a \in \partial(\{z \in \mathbb{C} | \text{Im}(z) > 0\})$ . By the maximum modulus principle, we have for all z in our domain

$$\begin{split} |F(z)| &\leq 1 \\ \Leftrightarrow \left| \frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}} \frac{z - \overline{z_0}}{z - z_0} \right| \leq 1 \\ \Leftrightarrow \left| \frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}} \right| \leq \left| \frac{z - z_0}{z - \overline{z_0}} \right|, \end{split}$$

As desired. Taking the limit as  $z_0$  approaches z, the expression becomes

$$\frac{|f'(z)|}{|2\mathrm{Im}\ f(z)|} \leq \frac{1}{|2y|}.$$

By our restriction y > 0, Im  $f(z) \ge 0$ , we can say

$$\frac{|f'(z)|}{\operatorname{Im} f(z)} \le \frac{1}{y},$$

as desired.