Convolution: Given functions of real variable f, K we define:

$$(f * K)(x) = \int f(t)\, K(x-t)\, dt$$

K: kernel function. Think of $(f * K)$ as result of plugging f into an integral transform given by K.

e.g. $K, f : \mathbb{R} \to \mathbb{R}$ then domain of integration $\mathbb{R}$.

$K, f : \mathbb{C} \to \mathbb{C}$, then over $\mathbb{C}$, etc.

Examples of <u>nice facts</u> about convolution: (Wheedon - Zygmund)

① $f \in L^p(\mathbb{R})$, $K \in C_0^m(\mathbb{R})$: comp. supp. functions with m continuous deriv.

$$\Rightarrow \quad f * K \in C_0^m, \quad \text{with} \quad \frac{d^i}{dx^i}(f * K) = f * \frac{d^i}{dx^i} K$$

$i \leq m$

② $K \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} K = 1$, then

defining $K_\varepsilon := \frac{1}{\varepsilon} \cdot K\left(\frac{x}{\varepsilon}\right)$, then $f * K_\varepsilon \to f$ as $\varepsilon \to 0$

(in L^p-norm if $f \in L^p(\mathbb{R})$)

Imagine $K = \mathbb{1}_{[0,1]}$ then K_ε has shrinking support. Larger values

"Approximation to Identity"

③ Connections to Fourier analysis. $\hat{}$: Fourier transform

(Tools used to prove, for example, that then $\widehat{(f*g)} = \hat{f} \cdot \hat{g}$

$C_0^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, $p \in [1, \infty)$

Use (2) to prove Fourier inverse formula

<u>Point</u>: Cauchy's integral formula is convolution with $K(z) = \frac{1}{z}$. (or $\frac{1}{2\pi i z}$)

on circle centered at origin: $\int K = 2\pi i$, so normalize

Better: supported on unit circle.