ODDS & ENDS:

(1) Last week, we were proving modulus principle w/ ∞-boundaries.

Simplified proof is included in notes.

(2) Additional hypothesis needed in problem 2 of this week's pset.

The set E is the closure of a region (open, conn. set)

(3) Question about generalizations of max mod. principle:

unit disk, $f(0) = 0$: Schwarz' lemma $|f(z)| \leq |z|$ why these functions?

$f(z) \leq 1$ on boundary

strip (infinite horizontal) : need $|f(z)| \ll e^{e^{c \cdot Re(z)}}$ $0 \leq c < 1$.

$f(z) \leq 1$ on boundary

Answer: Depends largely on geometry of domain. Hierarchy of functions with certain growth property:

c , $\log(z)$, z^k , e^z , ... e^{e^z} ...

constant

Analyze above two examples from this perspective

$\log(\log(z))$

etc.

same here if $k \in \mathbb{Q} \setminus \mathbb{Z}$

Issues with these functions over $\mathbb{C}$ as not entire, single-valued functions in nbhd of 0

(4) Please keep reading.

Joys of doing math:

- hw problems
- reading and understanding pf.

asking further questions

Back to proof of Cauchy's thm(s). First, two lemmas we've met before:

Lemma 1: γ: smooth (closed) curve. f continuous on γ

then $F(z) = \int_\gamma \frac{f(w)}{(w-z)}\,dw$ defines analytic function on $\mathbb{C} - \{\gamma\}$.

(proved this early in course. true if we replace $(w-z)$ by $(w-z)^m$ as well.)
used partial fractions to prove F continuous ...
or quote general result on diff. under integral sign. More generally...

Lemma 2: Ω: open set. γ: smooth (closed) curve.

$\phi : \{\gamma\} \times \Omega \longrightarrow \mathbb{C}$ continuous.

$G(z) = \int_\gamma \phi(w,z)\,dw$ defines continuous function.

if $\frac{\partial \phi}{\partial z}$ is continuous on $\{\gamma\} \times \Omega$, then G analytic and

$$G'(z) = \int_\gamma \frac{\partial \phi}{\partial z}(w,z)\,dw$$

(pick $\phi(w,z) = \frac{f(w)}{w-z}$ on $\{\gamma\} \times (\mathbb{C} \setminus \{\gamma\})$ to recover previous result.)

pf. followed by analogous result for Riemann integrals.

Maybe discuss pf a bit more at the end. See Conway IV. Prop. 2.1.
fairly straightforward δ-ε pf.

Strategy: Prove general version of Cauchy integral formula, using

$\phi(w,z) = \frac{f(w) - f(z)}{w - z}$, then Cauchy's thm: $\int_\gamma f\,dz = 0$ will be corollary.

Word about logic: CIF / Cauchy's thm on disk $\Rightarrow$ local props $\Rightarrow$ general versions of thm.

Cauchy integral formula: Ω open, $f : \Omega \to \mathbb{C}$ analytic

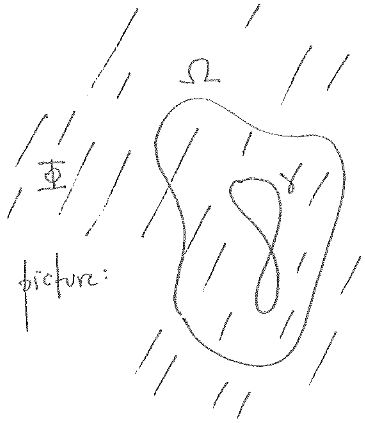
(3)

γ: closed smooth curve, s.t. $n(\gamma; z) = 0 \quad \forall z \in \mathbb{C} \setminus \Omega$

then given $a \in \Omega \setminus \{\gamma\}$,

$$n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - a} \, dw$$

picture:



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pf: Define $g(z) : \mathbb{C} \to \mathbb{C}$ by:

$$g(z) = \begin{cases} \int_\gamma \phi(w, z) \, dw & \text{if } z \in \Omega \\ \int_\gamma \frac{f(w)}{w - z} \, dw & \text{if } z \in \Phi \end{cases}$$

$\phi(w, z) = \frac{f(w) - f(z)}{w - z}$ if $w \neq z$

$= f'(z)$ if $w = z$

(continuous)

$z \mapsto \phi(w, z)$ analytic for fixed w.

Φ: $\{ z \in \mathbb{C} : n(\gamma; z) = 0 \}$ (open set)

Note g is well-defined since, on $\Omega \cap \Phi$,

$$\int_\gamma \frac{f(z)}{w - z} \, dw = f(z) \cdot \int_\gamma \frac{1}{w - z} \, dw = f(z)\, n(\gamma, z) \cdot 2\pi i = 0$$

since $z \in \Phi$.

Also note $\Omega \cup \Phi = \mathbb{C}$ by assumption.

Lemmas $\Rightarrow$ g is entire. Analyze growth properties.

$$\lim_{z \to \infty} g(z) = \lim_{z \to \infty} \int_\gamma \frac{f(w)}{w - z} \, dw$$

since $\Phi \ni \{\infty\}$: $n(\gamma; z) = 0$ on unbounded component containing $\{\infty\}$.

$= 0$ since $f(w)$ bounded on $\{\gamma\}$

$(w - z)^{-1} \to 0$ uniformly for $w \in \{\gamma\}$

$\Rightarrow g$ bounded $\Rightarrow g$ constant by Liouville's thm. $\Rightarrow g \equiv 0$ (since limit $= 0$)

in particular, on Ω, $\int_\gamma \phi(w, z)\, dw \equiv 0$, i.e.

$$\int_\gamma \frac{f(w)}{w-a}\, dw = f(a) \int_\gamma \frac{dw}{w-a} = f(a) \cdot n(\gamma; a) \cdot 2\pi i. \quad \text{as desired.}$$

Extend result to $\gamma = \gamma_1 + \cdots + \gamma_m$ formal sum of closed, piece-wise smooth curves using identical argument, with

$$\Phi = \{ z \mid n(\gamma_1; z) + \cdots + n(\gamma_m; z) = 0 \}$$

(requiring $n(\gamma_1; z) + \cdots + n(\gamma_m; z) = 0$ on $\mathbb{C} \setminus \Omega$.)

Statement: $$f(a) \cdot \sum_{j=1}^{m} n(\gamma_j; z) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(w)}{w-a}\, dw$$

if $a \in \Omega \setminus \{\gamma\}$

$\bigcup_j \{\gamma_j\}$

Finally Cauchy's thm: Ω open, f: analytic on Ω

γ as above. Then $\int_\gamma f = 0$.

Pf: Use $f(z)/z-a$ in place of f in previous theorems

Return to discussion of topology.