Last time, proved residue thm:

f analytic function except for isolated singularities $\{a_j\}$ in Ω : open, conn.

$\gamma \approx 0$ in Ω, $\{a_j\} \notin \{\gamma\}$

then
$$\frac{1}{2\pi i}\int_\gamma f\,dz = \sum_j n(\gamma, a_j)\, \underbrace{R_j}_{\text{residue of } f \text{ at } a_j.}$$
(no conv. issue in sum, since $n(\gamma, a_j) = 0$ for almost all j.)

Example:
$$\int_\gamma \frac{e^{2z}}{(z-1)^3} = n(\gamma;1)\, R_1 \cdot (2\pi i)$$
so by Residue thm

γ: any smooth closed curve

Only singularity is pole of order 3 at $z=1$.

$n(\gamma;1)$ depends on curve γ, but we can compute R_1.

Formula for residue:
$$a_{-1} = \frac{1}{(3-1)!}\frac{d^2}{dz^2}(z-1)^3 \cdot \frac{e^{2z}}{(z-1)^3}\Bigg|_{z=1}$$
$$\left(\text{i.e. } \frac{1}{(h-1)!}\frac{d^{h-1}}{dz^{h-1}}\left[(z-a_j)^h f(z)\right]\Bigg|_{z=a_j}\right)$$
$$= 4e^2 \cdot \frac{1}{(3-1)!} = 2e^2.$$

So answer: $4\pi i \cdot e^2 \cdot n(\gamma;1)$.

(HW. p.154 #1,2 postponed until next Friday)

If f has a simple pole (i.e. pole of order 1) then

$$a_{-1} = \lim_{z \to a_j} (z - a_j) f(z)$$

In particular if f rational function or any g, h analytic except for... $f = g/h$, then with h having simple 0 at a_j not canceled by g

$$\lim_{z \to a_j} (z - a_j) \frac{g(z)}{h(z)}$$

$$= \underbrace{\lim_{z \to a_j} g(z)}_{g(a_j)} \cdot \underbrace{\lim_{z \to a_j} \frac{(z - a_j)}{h(z)}}_{\text{L'Hôpital's rule: } \frac{1}{h'(a_j)}} = \frac{g(a_j)}{h'(a_j)}$$

Example: $f(z) = \dfrac{\sin z}{e^{2z}(z-3)}$ then $a_{-1}(f)$ in nbhd of 3 is

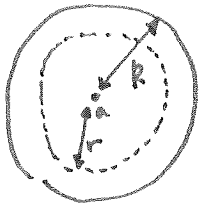
$$\frac{\sin 3}{e^{2 \cdot 3}} = \boxed{\frac{\sin 3}{e^6}}$$

since $\dfrac{d}{dz}\left(e^{2z}(z-3) \right) = 2e^{2z} \cdot (z-3) + e^{2z}$

(also see this by thinking about power series expansions...)

Return to our discussion of power series / Laurent series power series : f analytic on $B(a,R)$. Pick $r<R$, $\gamma = re^{it}$, $t\in[0,2\pi]$

Fix $z \in B(a,r)$.



$$f(z) \underset{\text{CIF}}{=} \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi$$

$$\underset{\text{WANT}}{=} \sum_{n=0}^{\infty} \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi-a)^{n+1}}\,d\xi}_{f^{(n)}(a)/n!}\,(z-a)^n$$

Play games with fractions:

$$\frac{1}{\xi - z} \underset{\substack{\uparrow \\ \text{algebra}}}{=} \frac{1}{(\xi-a)}\cdot\frac{1}{1-\left(\frac{z-a}{\xi-a}\right)} \underset{\substack{\uparrow \\ \text{geom. series}}}{=} \frac{1}{(\xi-a)}\sum_{n=0}^{\infty}\frac{(z-a)^n}{(\xi-a)^n}$$

formally, we're done, but need to justify interchange of integration over γ / summation over n

$$F_N = \sum_{n=0}^{N} \frac{f(\xi)(z-a)^n}{(\xi-a)^{n+1}} \qquad z, a \text{ fixed.}$$

- show $\displaystyle\int_\gamma \lim_N F_N = \lim_N \int_\gamma F_N$ if $F_N \to F$ uniformly for all $\xi \in \{\gamma\}$.

- prove $F_N \to F$ uniformly for all $\xi \in \{\gamma\}$.

Second bullet is just: $$\left| \frac{f(\xi)}{(\xi-a)} \frac{(z-a)^n}{(\xi-a)^n}\right| \le \frac{M}{r}\underbrace{\frac{|(z-a)|^n}{r^n}}_{\frac{|z-a|}{r} < 1 \text{ if } z\in B(a,r)} \quad \checkmark.$$

(Weierstrass M-test)

Laurent series (first principles) $\{ z_n \mid n = 0, \pm 1, \pm 2, \dots \}$

doubly infinite sequence

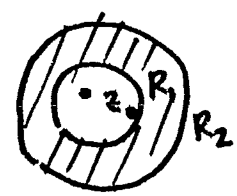
we say $\sum_{n=-\infty}^{\infty} z_n$ is absolutely convergent if $\sum_{n=0}^{\infty} z_n$, $\sum_{n=1}^{\infty} z_{-n}$ are both absolutely convergent

and then define $\sum_{n=-\infty}^{\infty} z_n := \sum_{n=0}^{\infty} z_n + \sum_{n=1}^{\infty} z_{-n}$.

(similarly, say convergence is uniform if both pieces converge uniformly on a set S.)

want to focus on absolutely convergent series here.

Thm: f analytic on annulus centered at z_0 with radii $0 \le R_1 < R_2$



Set $a_n = \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(z)}{(z-z_0)^{n+1}} dz$

$r \in (R_1, R_2)$

Then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ (where convergence is absolute and uniform away from boundary of annulus: i.e. on annulus with radii r_1, r_2 s.t. $R_1 < r_1 < r_2 < R_2$.)

and this series rep'n is unique.

see p. 184 of Ahlfors.

pf sketch: Given $z \in \mathrm{Ann}(z_0, R_1, R_2)$

then find r_1, r_2 with $R_1 < r_1 < r_2 < R_2$ s.t. $z \in \mathrm{Ann}(z_0, r_1, r_2)$

Consider cycle $\gamma = C(z_0, r_2) - C(z_0, r_1) \sim 0$ in $\mathrm{Ann}(z_0, R_1, R_2)$

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi \qquad \gamma = \underset{C(z_0,r_2)}{\underset{\shortparallel}{\gamma_2}} - \underset{C(z_0,r_1)}{\underset{\shortparallel}{\gamma_1}} \sim 0$$

$$= \underbrace{\frac{1}{2\pi i} \int_{C(z_0,r_2)} \frac{f(\xi)}{\xi - z}\, d\xi}_{f_2(z)}$$

since $n(\gamma, z) = 1$ for all points z in $Ann(r_1, r_2)$

$$- \underbrace{\frac{1}{2\pi i} \int_{C(z_0,r_1)} \frac{f(\xi)}{\xi - z}\, d\xi}_{f_1(z)}$$

(note winding number for cycle satisfies $n(\gamma_1 + \gamma_2, z) = n(\gamma_1, z) + n(\gamma_2, z)$)

Compute power series for f_1, f_2, add them together.

Recall that $f_2(z)$ defines an analytic function for $|z - z_0| < r_2 < R_2$

But r_2 arbitrary, so $f_2(z)$ defines analytic function on $B(z_0, R_2)$.

(Cauchy's thm says integrals over r_2, r_2' are equal for any $R_1 < r_2 < r_2' < R_2$)*

(so has power series expansion in positive powers)

Similarly, $f_1(z)$ defines analytic function for z with $|z - z_0| > r_1 > R_1$

so for any z with $|z - z_0| > R_1$ since r_1 arbitrary again.

then map image outside R_1 to disk:

$\xi \mapsto z_0 + 1/\xi'$

$z \mapsto z_0 + 1/z'$, and we're done...

* as long as $z \notin C(z_0, r_2)$ or $C(z_0, r_2')$