Rouché's thm (general version)

$$|f(z)+g(z)| < |f(z)|+|g(z)| \quad \text{on } \gamma = \text{circle of radius } R$$

(for simplicity)

f, g ~~holomorphic~~ meromorphic on $\Omega \supseteq \gamma$

then $Z_f - P_f = Z_g - P_g$.

Z_f : zeros of f in γ

Z_g : — " — g

P_f : poles of f in γ

P_g : — " — g in γ

pf : same idea. divide! and use Both interpretations of argument principle.

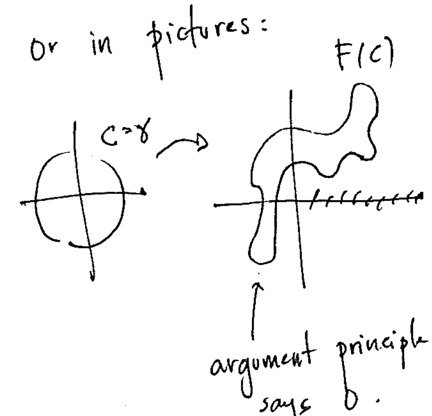
$$\left| f/g + 1 \right| < \left| f/g \right| + 1. \quad \text{on } \gamma$$

if f/g real, then contradiction. So $F = f/g$ maps γ to $\mathbb{C} \setminus [0,\infty)$

$\Rightarrow$ has well defined logarithm.

so
$$0 = \frac{1}{2\pi i}\int_\gamma \frac{(f/g)'}{(f/g)}\,dz = (Z_f - P_f) - (Z_g - P_g)$$

or in pictures:



Next : Residue thm. for evaluating definite integrals.

did guided first example in homework.

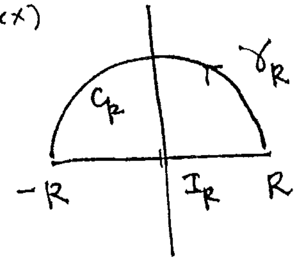
finite integrals : $\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$

$z = e^{i\theta}$.

indefinite integrals.

Example : $\displaystyle\int_{-\infty}^{\infty} \underbrace{\frac{x^2+3}{(x^2+1)(x^2+4)}}_{f(x)}\, dx \;.\; = I$

Consider closed contour :



write $\gamma_R = \underbrace{C_R}_{\substack{\text{semi-circle}\\ \text{of radius}\\ R}} + \underbrace{I_R}_{\substack{\text{real line}\\ \text{between}\\ [-R,R]}}$.

$\displaystyle\int_{\gamma_R} f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{I_R} f(x)\,dx$, then take limits of both sides as $R \to \infty$.

$I_R \to I$.

and hope $\displaystyle\int_{C_R} \to 0$.

First compute $\displaystyle\int_{\gamma_R} f(z)\,dz$, using residue thm.

Residues of $f(z)$ = $\pm i, \pm 2i$, so if $R > 2$, then we include both residues $\underline{i, 2i}$ in γ_R.

These are simple poles. (if $f = g/h$, then if z_0 is simple pole, residue is $g/h'\big|_{z=z_0}$.)

$f = \dfrac{z^2+3}{z^4+5z^2+4}$ so residue at $\underline{i}$: $\dfrac{-1+3}{4\cdot(i)^3+10i} = \dfrac{1}{3i}$

at $\underline{2i}$: $\dfrac{-4+3}{-32i+20i} = \dfrac{1}{12i}$

so for $R > 2$, $\displaystyle\int_{\gamma_R} f(z)\,dz = \left(\frac{1}{3i} + \frac{1}{12i}\right)\cdot 2\pi i = 5\pi/6$.

For $\displaystyle\int_{C_R} f(z)\,dz$: if $|z| = R$ then estimate $f(z)$. $|z^2+3| \le R^2+3$

$|z^2+1| = |z^2-(-1)|$

$\ge R^2 - 1$

similarly $|z^2+4| \ge R^2-4$

so $\displaystyle\int_{C_R} f(z)\,dz \le \int_0^{\pi} |f(z(t))\, z'(t)|\, dt$

$$\leq \underbrace{\pi}_{\int dt} \cdot \underbrace{R}_{|z'(t)|} \cdot \frac{R^2+3}{(R^2-1)(R^2-4)} = \pi \frac{R^3+3R}{R^4-5R^2+4} \to 0 \quad \text{as } R \to \infty$$

clear that integral $\to 0$ whenever we have rational function $f = g/h$ with $\deg g \leq \deg h - 2$

(Note that real improper integral itself converges iff this same inequality is satisfied) (Better: $\deg(h) - \deg(g) \geq 2$)

Example 2: $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx \qquad (a>0)$$

Consider contour integral: $$\int_{\gamma_R} \frac{e^{iz}}{z^2+a^2}\,dz$$ since $\operatorname{Re}(e^{ix}) = \cos x$.

(hence for I_R, $\operatorname{Im}(e^{ix}) = \sin x$ which is odd, while denom. even, so integral of $\frac{\operatorname{Im}(e^{ix})}{x^2+a^2}$ will vanish on I_R.)

Only singularity of $\frac{e^{iz}}{z^2+a^2}$ in upper-half plane is at $z = \pm ai$, which will be in γ_R if $R > a$.

Residue: $e^{-a}/2ai$ so $$\int_{\gamma_R} f(z)\,dz = e^{-a}\cdot\frac{\pi}{a}.$$

knew this would happen since $\mathbb{R}$ is real

Now on $|z| = R$, $|e^{iz}| = |e^{ix-y}|$ if $z = x+iy$

≤ 1 (the advantage of using e^{iz} instead of $\cos z$)

so $$\left|\int_{C_R} f(z)\,dz\right| \leq \pi\cdot R\cdot\frac{1}{R^2-a^2} \to 0 \quad \text{as } R\to\infty$$

Example 3 : $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx$ Can compute residues. A little messy.

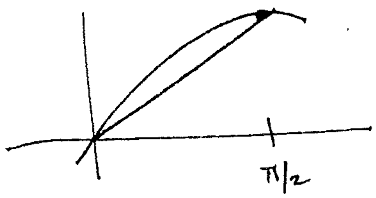
Estimating on C_R ? $\displaystyle\left| \frac{z^3}{(z^2+a^2)(z^2+b^2)} \right| \le \frac{M}{R}$, M: constant

so need estimate for $\displaystyle\int_{C_R} |e^{iz}|\,|dz|$. (our $|e^{iz}| = e^{-y} \le 1$ is too coarse.

on C_R, $y = R\sin\theta$ so we must estimate:

$$\int_0^{\pi} e^{-R\sin\theta} \cdot R\,d\theta$$

since $|dz| = R\,d\theta$.



$\sin\theta \geq \frac{2}{\pi}\theta$ on $[0, \pi/2]$

so $$\int_0^{\pi} e^{-R\sin\theta} R\,d\theta = 2\int_0^{\pi/2} e^{-2R\theta/\pi} R\,d\theta$$

$$= -\pi\, e^{-2R\theta/\pi}\Big|_0^{\pi/2} \le -\pi\,(e^{-1}-1) < \pi .$$

so on C_R, $\displaystyle\left|\int_{C_R} f(z)\,dz\right| \le \pi \cdot \frac{M}{R} \longrightarrow 0$ as $R \to \infty$.