Today: analytic functions.

Study continuous / diff. functions $f : \mathbb{C} \to \mathbb{C}$

(since $\mathbb{C} \simeq \mathbb{R}^2$ as metric space, all definitions and basic thms are same :

continuity: $\lim_{z \to a} f(z) = f(a)$ ← these limit def'ns are just ε-δ def'n with $\mathbb{C}$ absolute value.

diff.: $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists and call it $f'(a)$.

Assume functions defined on open set (every pt. has open nbhd (ball) in set) Ω

say that f is analytic on Ω if it is differentiable ~~at~~ all points in open set.
(also: holomorphic)

[Sometimes semantic distinction if want to stress that analytic = agrees with power series in an open nbhd.]
or continuously differentiable.

We'll say more about topology in a few classes (mostly reminders), for now explore basic properties of diff. and a few examples of analytic functions.

Note assumption that f defined on open set rules out

$f : \mathbb{R} \to \mathbb{C}$.

$\mathbb{R}$ not open... But this is just one variable calculus.

Write $f(t) = u(t) + iv(t)$

study u, v with calculus / $\mathbb{R}$.

All results like

① $(f+g)'(z) = f'(z) + g'(z)$

② $(fg)'(z) = (fg' + gf')(z)$

etc. rely only formal def'n of diff., cont. + basic properties of absolute value.

③ f differentiable $\Rightarrow$ f continuous

④ quotient rule ⑤ chain rule ...

Big difference: $f(z) = \mathrm{Re}(f(z)) + i\,\mathrm{Im}(f(z))$

$$\mathrm{Re}(f(z)) = \frac{f(z) + \overline{f(z)}}{2} \qquad \mathrm{Im}(f(z)) = \frac{f(z) - \overline{f(z)}}{2i}$$

Proposition: If f continuous, so are $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, $|f|$.

pf: $\lim_{z\to a} \overline{f(z)} = \bar{A}$ if $\lim_{z\to a} f(z) = A$. Why?

$$|f(z) - A| = |\overline{f(z) - A}| = |\overline{f(z)} - \bar{A}|.$$

This implies

$\frac{A+\bar{A}}{2}$.

all conclusions by sum, product rules for limits. i.e. $\lim_{z\to a} \mathrm{Re}(f) = \mathrm{Re}(A)$.

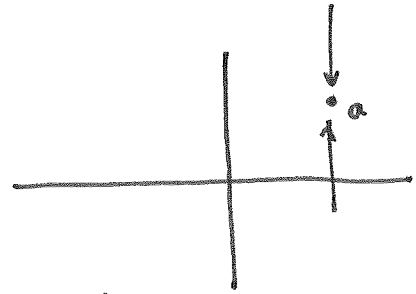
Q: if f diff., what can we say about $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, $|f|$?

(functions from $\mathbb{C} \to \mathbb{R}$)

problem: can't extract them from difference quotient for f.

Another approach: Analyze functions in general from $\mathbb{C} \to \mathbb{R}$ by taking difference quotient along various paths as $h \to 0$

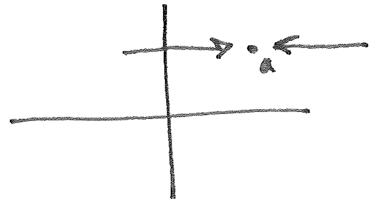
Path 1: h approaches along pure imag. path.



$$\lim_{\substack{h\to 0 \\ h:\text{real}}} \frac{f(a+ih) - f(a)}{ih} = \text{pure imag. \#}$$

since f real-valued so numerator in $\mathbb{R}$.

Path 2: h approaches along real path.



$$\lim_{\substack{h \to 0 \\ h:\text{ real}}} \frac{f(a+h) - f(a)}{h} = \text{purely real.}$$

Together these imply that $f'(a) = 0$. (Later we'll show this implies f is constant)

so $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, $|f|$ are not differentiable.

that worked well for $f: \mathbb{C} \to \mathbb{R}$, so try choosing same paths for $f: \mathbb{C} \to \mathbb{C}$ to obtain necessary conditions for differentiability:

<u>PATH 2</u>:

$$\lim_{\substack{h \to 0 \\ h \text{ real}}} \frac{f(a+h) - f(a)}{h} = \lim_{\substack{h \to 0 \\ h: \text{real}}} \frac{u(a+h) - u(a)}{h} + i \lim_{\substack{h \to 0 \\ h: \text{real}}} \frac{v(a+h) - v(a)}{h}$$

where

$f = u + iv$, $z = x + iy$

$u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$

$$= \left.\frac{\partial u}{\partial x}\right|_{z=a} + i \left.\frac{\partial v}{\partial x}\right|_{z=a} = \left.\frac{\partial f}{\partial x}\right|_{z=a}$$

Similarly, using <u>PATH 1</u>: $f'(a) = -i \dfrac{\partial f}{\partial y}$.

So must have $\dfrac{\partial f}{\partial x} = -i \dfrac{\partial f}{\partial y}$ or in terms of u, v, extracting real / imag. parts on each side ...

$$\ldots \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

"Cauchy - Riemann differential equations"

<u>Corollary</u>:

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \quad (*)$$

$$= \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \text{Jacobian of } u, v.$$

Corollary 2:

Let $\Delta u := \left(\dfrac{\partial^2 u}{\partial x^2}\right) + \left(\dfrac{\partial^2 u}{\partial y^2}\right)$. Δ: Laplacian diff op $= \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

If u, v have continuous first partial derivatives, then mixed partials [redacted] are equal, so $\Delta u = 0$ by C-R equations

Similarly $\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$. Functions which satisfy $\Delta f = 0$ are said to be "harmonic"

Notes: ① In preview lecture, we said that if f differentiable, then we will prove that it has derivatives of all orders, so in particular first partials will be continuous.

② Interpretation of $|f'(z)|^2$ as Jacobian is similar to multivariable real calculus/ Also harmonic functions are nice class of functions of real variable (say of $\mathbb{R}^2$): locally expressible as power series infinitely differentiable, …

Thm* Cauchy-Riemann equations are sufficient to guarantee differentiability. *: add additional hypothesis momentarily (not quite true as stated)

Pf: Write $f = u + iv$, $z = x+iy$ with $h+ik \to 0$ (h,k real)

Consider
$$u(x+h, y+k) - u(x,y) = u(x+h,y+k) - u(x,y+k) + u(x,y+k) - u(x,y)$$

aim: RHS $= \frac{\partial u}{\partial x}\cdot h + \frac{\partial u}{\partial y}\cdot k + \varepsilon(h,k)$ error with $\left\{ \frac{\varepsilon(h,k)}{h+ik} \to 0 \right.$ as $h+ik \to 0$.

Added Assumption: This statement requires continuity of first partial derivatives (*) $\longrightarrow$

prove claim using mean-value theorem.

same for $v(x+h, y+k) - v(x,y)$.

so $$f(z + (h+ik)) - f(z) = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)(h+ik) + \underbrace{\varepsilon + i\varepsilon}_{\text{errors.}}$$

used Cauchy-Riemann equations here

$$\Rightarrow \quad f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}. \quad ✓.$$

Non-example: $f(z) = \bar{z}$

$$\lim_{h \to 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h \to 0} \frac{\bar{h}}{h}.$$

Via real path: 1

via pure imag. path: -1

not differentiable!

Ahlfors describes useful formalism: Any function in x, y rewritten in terms of $z, \bar{z}$. Treat them as indep. vars...

$$\frac{\partial f}{\partial z} := \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \qquad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$

reasonable since $z = x + iy$ so
$$x = \tfrac{1}{2}(z + \bar{z})$$
$$y = -\tfrac{1}{2} i (z - \bar{z})$$

then for f differentiable, $\frac{\partial f}{\partial \bar{z}} = 0$

Mean value theorem applied to

$$u(x+h, y+k) - u(x, y+k) = \left.\frac{\partial u}{\partial x}\right|_{(x+h', y+k)} \cdot h$$

for some h' with $|h'| < |h|$.

Similarly

$$u(x, y+k) - u(x, y) = \left.\frac{\partial u}{\partial y}\right|_{(x, y+k')} \cdot k$$

Comparing, wanted $$u(x+h, y+k) - u(x,y) = \left.\frac{\partial u}{\partial x}\right|_{(x,y)} \cdot h + \left.\frac{\partial u}{\partial y}\right|_{(x,y)} \cdot k$$

$+$ error.

where
error : $$h \cdot \left[\left.\frac{\partial u}{\partial x}\right|_{(x+h', y+k)} - \left.\frac{\partial u}{\partial x}\right|_{(x,y)} \right]$$

$$+ \; k \left[\left.\frac{\partial u}{\partial y}\right|_{(x, y+k')} - \left.\frac{\partial u}{\partial y}\right|_{(x,y)} \right]$$

But $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ continuous $\Rightarrow \lim_{h+ik \to 0} \frac{\text{error}}{h+ik} = 0.$