Last time, showed diff. function $f: \mathbb{C} \to \mathbb{C}$ (or $\underset{\text{open}}{\Omega} \to \mathbb{C}$)

must satisfy Cauchy-Riemann eqns. ("2 paths argument")

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y} \qquad \left(\text{i.e. } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right)$$

if we write $f = u + iv$

Suppose Cauchy-Riemann eqn's are true.

Is f differentiable? (Almost. Need to add one assumption)

Mid proof: Analyzing difference quotient, whose numerator included:

$$\underbrace{u(x+h, y+k) - u(x, y+k)}_{\approx\, h \cdot \frac{\partial u}{\partial x}\big|_{(x,y)}} + \underbrace{u(x, y+k) - u(x,y)}_{\approx\, k \cdot \frac{\partial u}{\partial y}\big|_{(x,y)}}$$

← But off by error

To make precise: Use MVT

e.g. $u(x+h, y+k) - u(x, y+k) = h \cdot \left.\frac{\partial u}{\partial x}\right|_{(x+h', y+k)}$ for some h' in interval $(x, x+h)$

$$\begin{aligned} \text{so error from } u\text{'s} = h \cdot \left[\left.\frac{\partial u}{\partial x}\right|_{(x+h', y+k)} - \left.\frac{\partial u}{\partial x}\right|_{(x,y)} \right] \\ + k \cdot \left[\left.\frac{\partial u}{\partial y}\right|_{(x, y+k')} - \left.\frac{\partial u}{\partial y}\right|_{(x,y)} \right] \qquad (*) \end{aligned}$$

$$\frac{\text{error from } u\text{'s}}{h+ik} \to 0 \quad \text{as} \quad h+ik \to 0 \quad \text{since} \quad \left|\frac{h}{h+ik}\right| \le 1, \ \left|\frac{k}{h+ik}\right| \le 1$$

AND we ASSUME that partials are continuous. (so that terms in brackets in $(*) \to 0$ as $h+ik \to 0$.)

Similar calculation for v's,

$$\lim_{h+ik\to 0} \frac{f(z+(h+ik)) - f(z)}{h+ik} = \lim_{h+ik} \left[h\cdot \frac{\partial u}{\partial x}\Big|_{(x,y)} + k\frac{\partial u}{\partial y}\Big|_{(x,y)} + i\left(h\cdot \frac{\partial v}{\partial x}\Big|_{(x,y)} + k\frac{\partial v}{\partial y}\Big|_{(x,y)}\right)\right] \Big/ h+ik$$

Now use C-R to make sense of this limit : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

So get upon substituting: $\frac{h+ik}{h+ik}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$

whose limit is now clear!

Cor (of easy direction): f diff., then $\Delta u = \Delta v = 0$.
$f = u+iv$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

(since $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$ Mixed partials are equal if v has continuous first partials so these cancel.)

So u, v are "harmonic"

- Could give alternate equivalence:

  f diff $\Longleftrightarrow$ u, v harmonic + satisfy C-R equations

- Very basic PDE with interesting applications, see for example Dirichlet problem in Ch. 6 of Ahlfors

Polynomials : stated previously that polys. in $z = x+iy$ are differentiable since $f(z) = c$, c: constant, $f(z) = z$ are diff. and we have sum, product rules.

—

Given location of roots of $f(z)$, what can be said about the roots of $f'(z)$? For functions $\mathbb{R} \to \mathbb{R}$, question doesn't have a good answer. (E.g. consider shifting parabola like $f(x) = x^2 - 1$ upward / downward may or may not have real roots, but derivative always has a root at $x = 0$)

Assumption : Fundamental Thm. of Algebra
(to be proved on p. 122 of Ahlfors upon developing complex integration thy.)

Thus any polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$ can be expressed in form $P(z) = a_n (z - \alpha_1) \cdots (z - \alpha_n)$

α_n : roots (or "zeros")

Thm (Lucas-Gauss) If zeros of $P(z)$ are contained in (convex) polygon in $\mathbb{C}$, then roots (zeros) of $P'(z)$ are contained in same convex polygon.

pf : Suffices to show that if zeros of $P(z)$ lie in half plane H, then zeros of $P'(z)$ lie in H as well. I.e. $z \notin H \Rightarrow P'(z) \neq 0$

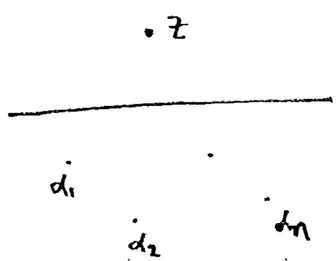
(After all, any convex polygon is just finite intersection of half planes)

Key idea : Show if $z \notin H$, $\dfrac{P'(z)}{P(z)} \neq 0$.

Note $\dfrac{P'(z)}{P(z)} = \dfrac{1}{z-\alpha_1} + \cdots + \dfrac{1}{z-\alpha_n}$ α_i roots.

(direct consequence of product rule. e.g. $n=2$, $P(z) = (z-\alpha_1)(z-\alpha_2)$,
$P'(z) = (z-\alpha_1) + (z-\alpha_2)$).

Easy example : H : lower half plane $\{ z \mid \mathrm{Im}(z) < 0 \}$



We see difference of imag. parts is always positive, so reciprocal

$\mathrm{Im}(z-\alpha_i)$

$\mathrm{Im}\left(\dfrac{1}{z-\alpha_i}\right)$ negative

so $\dfrac{P'(z)}{P(z)}$ has negative imaginary part.

General case is same idea.

Now line in $\mathbb{C}$ has parametric equation $z = a + bt$, $a, b \in \mathbb{C}$ fixed, $b \neq 0$. $t \in \mathbb{R}$ parameter

dividing $\mathbb{C}$ into half planes $\mathrm{Im}\left(\dfrac{z-a}{b}\right) < 0$ ← call this H.

$\mathrm{Im}\left(\dfrac{z-a}{b}\right) > 0$

Note $\mathrm{Im}\left(\dfrac{z-\alpha_k}{b}\right) = \mathrm{Im}\left(\dfrac{z-a}{b}\right) - \mathrm{Im}\left(\dfrac{a-\alpha_k}{b}\right)$.

If $z \notin H$, then $\nearrow$ >0 and $\nearrow$ <0 so $\mathrm{Im}\left(\dfrac{z-\alpha_k}{b}\right) > 0$

Hence $\operatorname{Im}\left(\frac{b}{z-d_k}\right) < 0$ so $\frac{b\,p'(z)}{p(z)} = \sum_{k=1}^{n} \frac{b}{z-d_k}$ has negative imag. part

so $p'(z) \neq 0$. ✓

Rational functions : Again differentiable if $Q(z) \neq 0$ in $\frac{P(z)}{Q(z)}$.

Derivative given by quotient rule.

* Places where $Q(z) = 0$ are called "poles" of rational function $\frac{P(z)}{Q(z)}$.

If we consider $R(z) = \frac{P(z)}{Q(z)} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$

(can check it is continuous at ∞ using our metric from stereographic proj.)

Set $R(\infty) = \lim_{z\to\infty} R(z)$, but this doesn't allow one to determine order of zero or pole at ∞.
i.e. multiplicity

Better : Change of coordinates $z \mapsto 1/z$
$\infty \mapsto 0$

Analyze function $R(\frac{1}{z})$ at $z = 0$.

Example : $R(z) = \frac{z^2-1}{z^3}$ has zeros at $1, -1$, (order 1)
pole of order 3 at $z = 0$

At $z = \infty$, analyze $R\left(\frac{1}{z}\right) = \frac{1/z^2 - 1}{1/z^3} = z - z^3$ so zero (of order 1) at ∞.

---

*: Note that we always want to consider $R(z) = \frac{P(z)}{Q(z)}$ in reduced form so that P, Q have no common factors.

Proposition: As a function on $\mathbb{C} \cup \{\infty\}$, $R(z)$ has equal number of zeros, poles (counted with multiplicity) and this equals $\max(\deg P, \deg R)$

pf: Bookkeeping at $z = \infty$ + Fundamental thm of algebra.

More interesting way to form analytic (i.e. holomorphic, differentiable) functions

Use Power Series.

(Read proof in Ahlfors 2.2.1 on completeness of $\mathbb{C}$: sequence converges iff it is Cauchy)

Just as with $\mathbb{R}$, series converge if ~~seq~~ sequence of partial sums converges.

There are stronger notions of convergence:

Say $\sum_{n=1}^{\infty} a_n$ <u>converges absolutely</u> if $\sum_{n=1}^{\infty} |a_n|$ converges.

(If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges by Δ-inequality)

Nice fact about absolutely convergent series: can rearrange order of terms without affecting the sum of the series (not so in general)

(Asked to prove this for next week's problem set)