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The symplectic Verlinde algebras and string K-theory

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Abstract

We construct string topology operations in twisted K-theory. We study the examples given by symplectic Grassmannians, computing $K^{\tau}_{*}(L\mathbb{H}P^{\ell})$ in detail. Via the work of Freed–Hopkins–Teleman, these computations are related to completions of the Verlinde algebras of $\mathrm{Sp}(n)$. We compute these completions, and other relevant information about the Verlinde algebras. We also identify the completions with the twisted K-theory of the Gruher–Salvatore pro-spectra. Further comments on the field-theoretic nature of these constructions are made.

Introduction

Much of the recent history of algebraic topology has been concerned with manifestations of ideas from mathematical physics within topology. A stunning example is Chas–Sullivan's theory of string topology [**4**], which provides a family of algebraic structures analogous to conformal field theory on the homology $H_*(LM)$ of the free loop space LM of a closed orientable manifold M [**17**]. The work of Chas and Sullivan started an entirely new field of algebraic topology, and led to papers too numerous to quote. Equally interesting as this analogy, however, is the fact that it is not quite precise: while the notion of conformal field theory is supposed to be completely self-dual, the string topology coproduct in H_*LM has no counit.

The inspiration for this paper came from two sources: one is the paper of Cohen and Jones [**6**], generalizing string topology to an arbitrary M-oriented generalized cohomology theory; the other is the work of Freed, Hopkins, and Teleman [**14**], which identified the famous Verlinde algebra of a compact Lie group G with its equivariant twisted K-theory. It follows that a completion of the Verlinde algebra is isomorphic to the (non-equivariant) twisted K-theory of LBG. In more detail, in [**14**], Freed–Hopkins–Teleman establish a ring isomorphism

$$ {}^{G}K^{*}_{\tau}(G) \cong V(\tau - h(G), G) \tag{1}$$

between the twisted equivariant K-theory of a simple, simply connected, compact Lie group G acting on itself by conjugation and the Verlinde algebra of positive energy representations of LG at level $\tau - h(G)$. It is well known that the Borel construction for the conjugation action $G \times_G EG$ is homotopy equivalent to LBG. So, using a twisted version of the Atiyah–Segal completion theorem due to Dwyer [**11**] and Lahtinen [**21**], we may conclude

$$ K^{*}_{\tau}(LBG) \cong V(\tau - h(G), G)^{\wedge}_{I}, \tag{2}$$

where the Verlinde algebra is completed at its augmentation ideal. Now a striking property of the Verlinde algebra is that it is a Poincaré algebra, which is the same thing as a 2-dimensional topological field theory (with no asymmetry). As far as we know, there is no analogous result involving ordinary homology: in some sense, while the K-theory information should be on the

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level of a modular functor of a conformal field theory, the field theory in homology should be on the level of the conformal field theory itself. From conformal field theory, one can produce finite models in characteristic 0 in the case of $N = 2$ supersymmetry, the A-model and the B-model. However, $N = 2$ supersymmetry corresponds to extra data beyond topology (Calabi–Yau structure). (Indeed, Fan, Jarvis, and Ruan [**12**] give a construction of A-module TQFT, coupled with compactified gravity, in the case of Landau–Ginzburg model orbifolds, which is related to the Calabi–Yau case, using Gromov–Witten theory. We return to this in the concluding remarks.) Is it then possible that, by considering an analogue of string topology on twisted K-theory, we could mimic, using string topology constructions, some of the properties of the Verlinde algebra, and in particular, perhaps, construct a self-dual topological field theory in some sense? The purpose of this paper was to investigate this question.

What we found was partially satisfactory, partially not. First of all, it turns out that twisted K-theory of loop spaces is quite difficult to compute, even in very simple cases. We originally thought that the question may be easier to tackle for quotients of symplectic groups (such as quaternionic Grassmannians), because of their relatively sparse homology. This led us to specialize to the symplectic case (it is, of course, only an example; analogous discussions should exist for other compact Lie groups). However, it turns out that the question is quite hard, and relatively little could be said beyond the case of projective spaces using our methods. Even for $\mathbb{H}P^\ell$, where one has a collapsing spectral sequence, it appears that one can only describe the exact extensions by considering, indeed, the string topology product.

Exploring the connection with the Verlinde algebra turned out to be more of a success: indeed, we show that the product in the Verlinde algebra is connected with the string topology product in the twisted K-theory of free loop space. A key step is the investigation of Gruher and Salvatore [**18**, **19**], who introduced spaces interpolating between the loop spaces of finite and infinite Grassmannians. On the other hand, it turned out that the coproduct is *not* related in the same way to the Verlinde algebra coproduct, and in fact exhibits the same asymmetry as elsewhere in string topology. In particular, the 'genus stabilization' string topology operation is 0, while it is always injective for the Verlinde algebra. The genus stabilization T is interesting for the following reason. Godin has recently shown [**17**] that $H_*(LM)$ admits the structure of a 'non-counital homological conformal field theory': this means that $H_*(LM)$ is an algebra over the homology $H_*(\mathcal{S})$ of the Segal–Tillmann surface operad, built out of the classifying spaces of mapping class groups. In [**24**], it is shown that for a topological algebra A over $\mathcal{S}$, the group completion of A is an infinite loop space. This group completion involves inverting T. In the string topology setting, this trivializes the algebra, since $T = 0$. On the other hand, one can show, for example, that a K-module whose 0-homotopy is the Verlinde algebra with T inverted admits the structure of an E_∞-ring spectrum (this will be discussed in a subsequent paper). We speculate in the concluding remarks of this paper that perhaps a Gromov–Witten type construction can lead to topological field theories in this context, bridging the gap between both contexts.

Computationally, we determine the twisted K-homology string product and coproduct for quaternionic projective spaces, exhibiting that these structures contain, in a cute way, more information than the corresponding structures in ordinary homology. Also, one gets the sense that the connection with the Verlinde algebra makes the twisted K-theory construction somehow 'smaller' than the untwisted case (by virtue of the d_3 differential in the twisted Atiyah–Hirzebruch spectral sequence (AHSS)), although the answer is not finite.

The present paper is organized as follows. In Section 1, we start, as a 'warm-up case', the computation of twisted K-theory of the free loop spaces of symplectic projective spaces and the related Gruher–Salvatore spaces. We see, in particular, that even here, the free loop space is tricky, and to resolve it further, the string topology structure is needed. This, in some sense, serves as a motivation for the remainder of the paper. In Section 2, we attempt to extend this calculation to the case of symplectic Grassmannians of $\mathrm{Sp}(n)$ with $n > 1$.

We shall see that the situation there is still much more complicated, and raises purely algebraic questions about the symplectic Verlinde algebras, for example the structure of their associated graded rings under the Atiyah–Hirzebruch filtration. These and related algebraic questions will be treated in Sections 3 and 4. In Section 3, we specifically consider the case of Verlinde algebras of representation level 1, where more precise information can be obtained. In Section 4, we consider the general case. Our results include a complete explicit computation of the Douglas–Braun number, the completion of the symplectic Verlinde algebras, a complete computation of T for the case $n = 1$, and an estimate (and precise conjecture) for the $n > 1$ case. In Section 5, we finally return to string topology, constructing the string product in twisted K-theory, and computing the string topology ring structure on the twisted K-theory of $L\mathbb{H}P^\ell$, and as a result, we determine the extensions in its additive structure. We shall see that in some sense, the ring structure is more non-trivial and interesting than in homology. In Section 6, we consider the string coproduct, show the failure of stabilization, and determine the string coproduct for quaternionic projective spaces (with a non-trivial result). Finally, in Section 7, we discuss the comparison of the product with the Verlinde algebra product via Gruher–Salvatore spaces, and the failure of analogous behaviour in the case of the coproduct. Section 8 contains concluding remarks, including the speculations on Gromov–Witten theory. In the appendix, we review very briefly some foundational material needed in this paper.

1. *The case* $n = 1$

In this paper, we focus on symplectic groups. Denote by $G_{\mathrm{Sp}}(\ell, n)$ the symplectic Grassmannian of n-dimensional $\mathbb{H}$-submodules of $\mathbb{H}^{\ell+n}$, where $\mathbb{H}$ is the algebra of quaternions. By $\mathrm{Sp}(n)$, we shall mean its compact form, which is the group of automorphisms of the $\mathbb{H}$-module $\mathbb{H}^n$ which preserve the norm. Of course, there is a canonical map $G_{\mathrm{Sp}}(\ell, n) \to B\mathrm{Sp}(n)$. Consequently, we can induce K-theory twistings on $LG_{\mathrm{Sp}}(\ell, n)$ from K-theory twistings on $LB\mathrm{Sp}(n)$. Here L denotes the free loop space. Twistings are classified by $H^3(?, \mathbb{Z})$. We have $H^3(LB\mathrm{Sp}(n), \mathbb{Z}) \cong H^3_{\mathrm{Sp}(n)}(\mathrm{Sp}(n), \mathbb{Z})$, where the right-hand side denotes Borel cohomology, and $\mathrm{Sp}(n)$ acts on itself by conjugation. As reviewed in [**14**], this group is $\mathbb{Z}$, and there is a canonical generator ι such that the twisting $\tau = m\iota$ corresponds to level $m - n - 1$ lowest weight twisted representations of $L\mathrm{Sp}(n)$, when this number is positive (since $n + 1$ is the dual Coxeter number of $\mathrm{Sp}(n)$). This is the twisting we shall be considering here.

We begin with a concrete computation of $K^\tau_*(L\mathbb{H}P^\ell)$. We proceed in stages, beginning with a computation of the completion of the Verlinde algebra for $\mathrm{Sp}(1)$. Using this, we compute the twisted K-theory of an intermediate space $Y(\ell, 1)$ which, in combination with the loop product developed in Section 5, allows us to compute $K^\tau_*(L\mathbb{H}P^\ell)$.

The Verlinde algebra of twisting level $m \geqslant 3$ (loop group representation level $m - 2$) is isomorphic to

$$V_m = V(m, 1) = \mathbb{Z}[x]/\mathrm{Sym}^{m-1}(x), \tag{3}$$

where x is the tautological representation of $\mathrm{Sp}(1) = \mathrm{SU}(2)$ and the polynomials $\mathrm{Sym}^k(x)$ are defined inductively by

$$\mathrm{Sym}^0(x) = 1, \quad \mathrm{Sym}^1(x) = x, \quad x\,\mathrm{Sym}^k(x) = \mathrm{Sym}^{k+1}(x) + \mathrm{Sym}^{k-1}(x). \tag{4}$$

The augmentation ideal I of V_m is generated by $x - 2$. If we change variables to $y = x - 2$, and define $\sigma^{m-1}(y) = \mathrm{Sym}^{m-1}(y + 2)$, then the completion of V_m at I is the quotient of the power series ring $\mathbb{Z}[[y]]$:

$$(V_m)^\wedge_I = \mathbb{Z}[[y]]/\sigma^{m-1}(y). \tag{5}$$

A straightforward induction shows that $\sigma^{m-1}(0) = \mathrm{Sym}^{m-1}(2) = m$, so m lies in the ideal (y). Consequently, equation (5) implies

$$(V_m)^\wedge_I = \prod_{p|m} \mathbb{Z}_p[[y]]/\sigma^{m-1}(y). \tag{6}$$

We would like to know how many copies of $\mathbb{Z}_p$ appear in this completion. We will prove a generalization of the following result in Section 4. However, it is instructive to prove this special case immediately.

PROPOSITION 1. *Let p be a prime and let $p^i \| m$ (by which we mean that p^i is the p-primary component of m). Let*

$$\delta(p,m) = \begin{cases} (p^i-1)/2 & if\ p \neq 2, \\ 2^i - 1 & if\ p = 2. \end{cases}$$

Then the p-primary component of $(V_m)^\wedge_I$ is $(\mathbb{Z}_p)^{\delta(p,m)}$.

Note that $\delta(p,m)$ is the number of $2m$th roots of unity that have positive imaginary part and are also p-power roots of unity.

Proof of Proposition 1. The polynomials

$$\mathrm{Sym}^{m-1}(x) \tag{7}$$

are Chebyshev polynomials of the second kind applied to $2x$. This means that if we let ζ_m be the primitive mth root of unity, then the roots of (7) are

$$\zeta_{2m}^k + \zeta_{2m}^{-k}, \quad k = 1, \dots, m-1. \tag{8}$$

This means that if we denote by W a ring of Witt vectors at the prime p in which (7) splits, we obtain an injective homomorphism

$$V_m \otimes W \to \prod_{k=1}^{m-1} W[x]/(x - \zeta_{2m}^k - \zeta_{2m}^{-k}) \tag{9}$$

with finite cokernel. Completing this at the ideal $(x-2)$ will therefore additively give rise to a product of as many copies of W as there are numbers $k = 1, \dots, m-1$ such that

$$\zeta_{2m}^k + \zeta_{2m}^{-k} - 2 \tag{10}$$

has positive valuation. But now (10) is equal to

$$\zeta_{2m}^{-k}(\zeta_{2m}^k - 1)^2,$$

so (10) has positive valuation if and only if $\zeta_{2m}^k - 1$ does. It is standard that $\zeta_\ell - 1$ has positive valuation if and only if $\ell = p^i$ for some i. Indeed, sufficiency follows from the fact that $((x+1)^{p^i} - 1)/((x+1)^{p^{i-1}} - 1)$ is an Eisenstein polynomial with root $\zeta_{p^i} - 1$. To see necessity, if $\zeta - 1$ has positive valuation, so does $\zeta^p - 1$, so it suffices to consider the case when ℓ is not divisible by p. But then we have a field $\mathbb{F}_p[\zeta_\ell]$, in which $\zeta_\ell - 1$ is invertible, so its lift to W cannot have positive valuation. $\square$

It follows from the completion theorem that the twisted K-theory of a simply connected simple compact Lie group is isomorphic to the completion of the Verlinde algebra (at a twisting where the Verlinde algebra exists) at the augmentation ideal with a shift equal to dimension and is equal to 0 in dimensions of opposite parity, so in particular

$$K^0_\tau(L\mathbb{H}P^\infty) = 0,\ K^1_\tau(L\mathbb{H}P^\infty) \cong (V_m)^\wedge_I. \tag{11}$$

Next, we compute the twisted K-theory of $L\mathbb{H}P^\ell$. We shall see that the answer is actually quite complicated; this motivates the structure which we shall introduce later, and to which we shall partially need to refer to complete the calculation.

Demonstrating another theme of this paper, we shall also see that before considering $L\mathbb{H}P^\ell$, it helps to consider the 'intermediate' space

$$Y(\ell,1) = L\mathbb{H}P^\infty \times_{\mathbb{H}P^\infty} \mathbb{H}P^\ell. \tag{12}$$

Note that (12) has the homotopy type of a finite-dimensional manifold: an Sp(1)-bundle over $\mathbb{H}P^\ell$. We shall see in Section 2 that $Y(\ell,1)$ is orientable with respect to K-theory. Thus, its twisted K-homology and cohomology with the same twisting are isomorphic, with a shift in dimensions, which in this case is odd. For now, however, the most important property for us is that the twisted K-theory of $Y(\ell,1)$ is easier to determine. Let us begin with a definition. Let i be such that $p^i \| m$. Let ℓ be a positive integer. Let r be a positive integer and let p be a prime. Suppose further that $r \leqslant \ell + 1$ and

$$2^{j-1} - 1 < r \leqslant 2^j - 1$$

if $p = 2$ and

$$\frac{p^{j-1}-1}{2} < r \leqslant \frac{p^j-1}{2}$$

if $p > 2$ for some $j = 1, \ldots, i$ if $p = 2$. Then put

$$\epsilon(p,\ell,r) = \left\lceil \frac{2+\ell-r}{2^{j-1}} \right\rceil$$

if $p = 2$ and

$$\epsilon(p,\ell,r) = \left\lceil \frac{2(2+\ell-r)}{(p-1)p^{j-1}} \right\rceil$$

if $p > 2$. Let

$$\epsilon(p,\ell,r) = 0$$

in all other cases.

Theorem 2. *We have*

$$K_1^\tau Y(\ell,1) = K^0_\tau Y(\ell,1) = 0 \tag{13}$$

and

$$K_0^\tau Y(\ell,1) = K^1_\tau Y(\ell,1) = \bigoplus_{p,r} \mathbb{Z}/p^{\epsilon(p,\ell,r)}. \tag{14}$$

A tool that we will need is the Serre spectral sequence in twisted K-theory for a fibration $F \to E \to B$ with twisting $\tau \in H^3(F)$ (and pulled back to E). It takes the form $H^*(B; K^*_\tau(F)) \Rightarrow K^*_\tau(E)$.

Proof of Theorem 2. First, the equality between twisted K-homology and cohomology follows from the fact that $Y(\ell,1)$ is an odd-dimensional, K-orientable manifold. The canonical inclusion

$$Y(\ell,1) \longrightarrow L\mathbb{H}P^\infty$$

induces a map

$$V(m,1)^{\wedge}_{I} \longrightarrow K^1_\tau Y(\ell,1). \tag{15}$$

Furthermore, (15) is a map of $K^0(\mathbb{H}P^\infty) = \mathbb{Z}[[y]]$-modules. Clearly $y^{\ell+1}$ annihilates the target (since the right-hand side is actually a $K^0(\mathbb{H}P^\ell)$-module), so (15) induces a map of $\mathbb{Z}[[y]]$-modules

$$\mathbb{Z}[y]/(\sigma^{m-1}(y), y^{\ell+1}) \longrightarrow K^1_\tau Y(\ell,1). \tag{16}$$

We claim that (16) is actually an isomorphism. To this end, consider the map of fibration sequences

$$\begin{tikzcd} \mathrm{Sp}(1) \arrow[r, "="] \arrow[d] & \mathrm{Sp}(1) \arrow[d] \\ Y(\ell,1) \arrow[r] \arrow[d] & L\mathbb{H}P^\infty \arrow[d] \\ \mathbb{H}P^\ell \arrow[r] & \mathbb{H}P^\infty. \end{tikzcd} \tag{17}$$

Since $K^1_\tau(\mathrm{Sp}(1)) = \mathbb{Z}/m$, the induced map on E_2 terms of twisted K-cohomology Serre spectral sequences is a 3-fold suspension of

$$(\mathbb{Z}/m)[u,u^{-1}][y] \longrightarrow (\mathbb{Z}/m)[u,u^{-1}][y]/(y^{\ell+1}). \tag{18}$$

Here u is the Bott class. Clearly, the spectral sequences collapse, so (18) induces an isomorphism between associated graded objects to a (finite) filtration on (16), and hence (16) is an isomorphism.

Thus, we are reduced to computing the p-completion $R(m,\ell)$ of the left-hand side of (16). Let us assume that $p^i \| m$, with $i \geqslant 1$. The key point is to consider the Eisenstein polynomial

$$\Phi = \Phi_{p,m} = \prod_{\zeta}(y + 2 - \zeta - \zeta^{-1}),$$

where the product is over all p^ith roots of unity ζ with $\mathrm{Im}(\zeta) \geqslant 0$ such that ζ is not a p^{i-1}th root of unity. Then multiplication by Φ defines an embedding of filtered $\mathbb{Z}_p[y]$-modules

$$R(m/p,\ell) \longrightarrow R(m,\ell), \tag{19}$$

which, on the associated graded objects, is given by multiplication by p (since $\Phi = p \bmod y$). By induction on i, this then determines how multiples by p of the generators

$$1, y, \dots, y^{\delta(p,m/p)-1} \tag{20}$$

are represented in (18): the p-multiples of the generators (20) are in the same filtration degree, and are represented as p-multiples. Multiples of (20) by higher powers p^j are given by taking the image under the associated graded map of (19) of the p^{j-1}-multiples of the corresponding generator (20) in the associated graded object of the left-hand side of (19). Since the associated graded object of the cokernel of (19) is therefore annihilated by p, on the remaining generators

$$y^{\delta(p,m/p)}, \dots, y^{\delta(p,m)-1}$$

multiplication by p is simply represented by the reduction modulo p of Φ, which is

$$y^{\delta(p,m)-\delta(p,m/p)} = y^{p^{i-1}/2}.$$

Recording these extensions in closed form gives the statement of the theorem. $\square$

It turns out that actually fully determining the twisted K-theory of $L\mathbb{H}P^\ell$ is subtle, and seems to require full use of the string topology product, and even then, the extensions do not seem to follow as simple a pattern as in the case of $Y(\ell,1)$. A complete answer will be postponed to Section 5. From a mere existence of the product, however, we can now state the following theorem.

THEOREM 3. *There is a vanishing*

$$K_1^\tau(L\mathbb{H}P^\ell) = K_\tau^0(L\mathbb{H}P^\ell) = 0. \tag{21}$$

Further, there exists a decreasing filtration on $K_0^\tau(L\mathbb{H}P^\ell)$ such that the associated graded object is

$$E_0K_0^\tau(L\mathbb{H}P^\ell) = K_0^\tau(Y(\ell,1)) \otimes \mathbb{Z}[t]. \tag{22}$$

Similarly, there is an increasing filtration on $K_\tau^1(L\mathbb{H}P^\ell)$ such that the associated graded object is

$$E_0K_\tau^1(L\mathbb{H}P^\ell) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[t], K_0^\tau(Y(\ell,1))). \tag{23}$$

Proof. We have a canonical fibration

$$\Omega S^{4\ell+3} \longrightarrow L\mathbb{H}P^\ell \longrightarrow Y(\ell,1), \tag{24}$$

where the second map is the projection. Consider the associated spectral sequence in twisted K-homology. The E^2 term then is concentrated in even degrees, and hence the spectral sequence collapses. On the other hand, the second map in (24), as we shall see in Section 5, is a map of rings, and in fact the entire spectral sequence is a spectral sequence of rings. It then follows that $K_0^\tau L\mathbb{H}P^\ell$ is generated, as a ring, by two generators y and t where y is as in (16), and t is the generator of

$$K_0\Omega S^{4\ell+3} = \mathbb{Z}[t].$$

Further, we see that $K_0^\tau L\mathbb{H}P^\ell$ satisfies the relation $y^{\ell+1}$, and a relation which is congruent to $\sigma^{m-1}(y)$ mod (t). The first statement follows. The second statement now follows from the universal coefficient theorem. □

2. *On the case $n > 1$*

There are a number of spectral sequences we may attempt to use for calculating the twisted K-theory of $LG_{\mathrm{Sp}}(\ell,n)$ for $n>1$. Let $W(\ell,n)$ denote the space of symplectic n-frames in $\mathbb{H}^{\ell+n}$. The most promising spectral sequence seems to be the twisted K-theory Serre spectral sequence associated with the fibration

$$\Omega W(\ell,n) \longrightarrow LG_{\mathrm{Sp}}(\ell,n) \longrightarrow Y(\ell,n). \tag{25}$$

Here

$$Y(\ell,n) = LB\mathrm{Sp}(n) \times_{B\mathrm{Sp}(n)} G_{\mathrm{Sp}}(\ell,n). \tag{26}$$

The reason this is advantageous is that $Y(\ell,n)$ is a homotopy equivalent to a finite-dimensional manifold, namely, a fibre bundle over $G_{\mathrm{Sp}}(\ell,n)$ with fibre $\mathrm{Sp}(n)$ (although not a principal bundle). Additionally, the manifolds $Y(\ell,n)$ are orientable with respect to K-theory because of the following result.

Lemma 4. *The K-theory (or ordinary) homology or cohomology spectral sequences associated to the fibrations*

$$\mathrm{Sp}(n) \longrightarrow Y(\ell, n) \longrightarrow G_{\mathrm{Sp}}(\ell, n), \tag{27}$$

$$\mathrm{Sp}(n) \longrightarrow LB\mathrm{Sp}(n) \longrightarrow B\mathrm{Sp}(n) \tag{28}$$

collapse to their respective E^2 and E_2 terms.

Proof. The fibration (27) maps in an obvious way into (28) which induces an injection on E^2 and a surjection on E_2 terms, respectively, of the spectral sequences in question. Thus, it suffices to consider (28). For the same reason, it suffices to consider $n = \infty$ in (28). But for $n = \infty$, (28) is a fibration of infinite loop spaces with the maps infinite loop maps. Since (28) splits, it is therefore a product in this case, which implies the desired collapse. □

We may therefore expect to use Poincaré duality together with the canonical comparison map

$$Y(\ell, n) \longrightarrow LB\mathrm{Sp}(n) \tag{29}$$

to help calculate the twisted K-theory AHSS for $Y(\ell, n)$ (see the next section for an example).

Unfortunately, the twisted AHSS for $LB\mathrm{Sp}(n)$ is extremely tricky, even though we know its target by [**14**]. If this calculation can be done, though, we can solve the spectral sequence of (25) by the following discussion. First note that by the Eilenberg–Moore spectral sequence, $H^*\Omega W(\ell, n)$ is the divided power algebra on bottom generators in dimensions

$$4\ell + 2, 4\ell + 6, \ldots, 4\ell + 4n - 2.$$

Since the dimensions are all even, the AHSS for $\Omega W(\ell, n)$ must collapse (and besides, there is no twisting when $\ell > 1$ by connectivity). The spectral sequence is completely determined as the tensor product of the twisted AHSS for $Y(\ell, n)$ and $H^*\Omega W(\ell, n)$ by the following result.

Theorem 5. *For $\ell \geqslant n$, the twisted Serre spectral sequence of the fibration (25) is a spectral sequence of $H^*\Omega W(\ell, n)$-modules.*

Proof. We use the diagonal map from (25) to its product with itself for the module structure. On the product, we can take the twisting trivial on one factor, and equal to the given twisting on the other. Thus, it suffices to show that the *untwisted* K-theory cohomology Serre spectral sequence associated with the fibration (25) collapses to the E_2 term. Suppose that this is not the case. Since the E_2 term is torsion-free, the first non-trivial differential must also appear in the corresponding $K\mathbb{Q}$-cohomology Serre spectral sequence, and hence in the ordinary cohomology spectral sequence. So, we must prove that the ordinary rational cohomology Serre spectral sequence associated with (25) collapses.

To this end, first note that, for $\ell \gg n$, the ordinary cohomology Serre sequence associated with the fibration

$$\Omega W(\ell, n) \longrightarrow \Omega G_{\mathrm{Sp}}(\ell, n) \longrightarrow \mathrm{Sp}(n) \tag{30}$$

collapses. Indeed, (30) is a principal fibration (the next term to the right is $W(\ell, n)$), so the corresponding homology spectral sequence is a spectral sequence of $H_*\Omega W(\ell, n)$-modules, but the elements $H_*\mathrm{Sp}(n)$ are permanent cycles, since they have no possible target). Now consider the Serre spectral sequence in ordinary cohomology associated with the fibration

$$\Omega G_{\mathrm{Sp}}(\ell, n) \longrightarrow LG_{\mathrm{Sp}}(\ell, n) \longrightarrow G_{\mathrm{Sp}}(\ell, n). \tag{31}$$

We claim that this also collapses to E_2. Indeed, we may map this into the corresponding Serre spectral sequence (28). By Lemma 4, this collapses, so it suffices to show that the polynomial generators of $H^*(\Omega W(\ell,n),\mathbb{Q})$ are permanent cycles in the cohomology Serre spectral sequence associated with (31). For example, we may map (25) to the case of $n=\infty$, in which case we get

$$\Omega\mathrm{Sp}/\mathrm{Sp}(\ell) \longrightarrow LB\mathrm{Sp}(\ell) \longrightarrow LB\mathrm{Sp}\times_{B\mathrm{Sp}} B\mathrm{Sp}(\ell). \tag{32}$$

By mapping into (32) the case $\ell=0$, which is the fibration

$$\Omega\mathrm{Sp} \longrightarrow * \longrightarrow \mathrm{Sp}, \tag{33}$$

and taking ordinary homology Serre spectral sequences, we know that for (33), the exterior generators of $H_*\mathrm{Sp}$ transgress to the corresponding polynomial generators of the homology of the fibre. This implies that in (32), the exterior generators of $H_*\mathrm{Sp}$ of dimensions $4\ell+3, 4\ell+7,\dots$ transgress and the ones in dimensions $3,\dots,4\ell-1$ are permanent cycles. This shows that the E_∞ term of the homology Serre spectral sequence of (32) is $H_*\mathrm{Sp}(\ell)\otimes H_*B\mathrm{Sp}(\ell)$, concentrated on a horizontal line.

Now dualizing, we see that in the $H\mathbb{Q}^*$-cohomology Serre spectral sequence of (32), the polynomial generators of $H^*(\Omega\mathrm{Sp}/\mathrm{Sp}(\ell),\mathbb{Q})$ transgress. Hence, the same is true of the polynomial generators of $H^*(\Omega W(\ell,n),\mathbb{Q})$ in the rational cohomology Serre spectral sequence of (25). What we need to show is that the transgressions of the generators of $H^*(\Omega W(\ell,n),\mathbb{Q})$ to $H^*(Y(\ell,n),\mathbb{Q})$ are 0. To this end, we claim the following:

For $n\leqslant \ell$, the map induced on homology by the natural inclusion $Y(\ell,n)=LB\mathrm{Sp}(n)\times_{B\mathrm{Sp}(n)} G_{\mathrm{Sp}}(\ell,n)\longrightarrow \mathrm{LBSp}\times_{B\mathrm{Sp}} B\mathrm{Sp}(\ell)$ factors through the map induced in homology by the natural inclusion $\mathrm{LBSp}(\ell)\longrightarrow LB\mathrm{Sp}\times_{B\mathrm{Sp}} B\mathrm{Sp}(\ell)$. (34)

Note that if (34) is proved, then the same statement is true on rational cohomology by the universal coefficient theorem. Then the images of the transgressions of the generators of $H^*(\Omega\mathrm{Sp}/\mathrm{Sp}(\ell),\mathbb{Q})$ map to 0 in $H^*(Y(\ell,n),\mathbb{Q})$, since they certainly map to 0 in $H^*(LB\mathrm{Sp}(\ell),\mathbb{Q})$ by the edge map theorem. This will conclude the proof of our theorem.

To prove (34), note that we can further compose with the map $LB\mathrm{Sp}\times_{B\mathrm{Sp}} B\mathrm{Sp}(\ell)\to LB\mathrm{Sp}$, since this map is injective on homology. But the natural inclusion

$$Y(\ell,n)=LB\mathrm{Sp}(n)\times_{B\mathrm{Sp}(n)} G_{\mathrm{Sp}}(\ell,n)\longrightarrow LB\mathrm{Sp}$$

certainly factors through the inclusion

$$LB\mathrm{Sp}(n)\longrightarrow LB\mathrm{Sp}. \tag{35}$$

However, note that this is induced in our setup by a map $\mathrm{Sp}(n)\to\mathrm{Sp}(\ell)$ induced by a quaternion-linear map $\mathbb{H}^n\to\mathbb{H}^\infty$ with the set of coordinates disjoint from those involved in the inclusion $\mathbb{H}^\ell\to\mathbb{H}^\infty$ involved in the inclusion $\mathrm{Sp}(\ell)\to\mathrm{Sp}$ which induces the map $LB\mathrm{Sp}(\ell)\to LB\mathrm{Sp}$ involved in the statement of (34); nevertheless, we have $n\leqslant\ell$, so the map (35) factors through a map induced by *some* inclusion $\mathrm{Sp}(\ell)\to\mathrm{Sp}$ induced by inclusion of coordinates, and any two such maps are homotopic as maps of group. The statement of (34) follows. □

3. *The representation level* 1 *symplectic Verlinde algebra*

Let us consider the case of twisting level $n+2$ for $\mathrm{Sp}(n)$ (representation level 1). The advantage is that in this case, we know by rank-level duality that the Verlinde algebra is isomorphic to the Verlinde algebra for $\mathrm{Sp}(1)$ at the same twisting level (that is, representation level n). More explicitly, the generators of the Verlinde algebra are the fundamental (that is, level 1) representations of $\mathrm{Sp}(n)$. To this end, we have a 'defining' representation of dimension $2n$, which we will for the moment denote by x. Then the other fundamental representations are

$$v_i = \Lambda^i(x) - \Lambda^{i-2}(x), \quad i = 2, \dots, n. \tag{36}$$

Note that there is a canonical contraction map $\Lambda^i \to \Lambda^{i-2}$. It turns out that $xv_i(x)$ for $i \geqslant 1$ (we put $v_0 = 1$, $v_1 = x$) contains v_{i-1} and v_{i+1} as subrepresentations, and their complement is an irreducible representation of level 2. This gives the relation

$$v_i x = v_{i+1} + v_{i-1}, \;\; i = 1, \dots, n-1, \quad v_n x = v_{n-1}. \tag{37}$$

There are more level 2 representations, but the corresponding relations are redundant.

We see immediately that (37) implies

$$v_i = \mathrm{Sym}^i(x), \tag{38}$$

where Sym^i are the polynomials from Section 1. The Braun–Douglas number $d(n) = d(n+2, n)$ is the greatest common divisor of the differences of dimension of the left- and right-hand sides of each relation (37). This number is contained in the augmentation ideal of the Verlinde algebra. Perhaps surprisingly, it turns out to be very small, making the completion trivial in most cases. We will, again, prove a generalization of the following theorem in Section 4.

THEOREM 6. *When $n \geqslant 2$, we have $d(n) = 2$ when $n = 2^\ell - 2$, and $d(n) = 1$ otherwise.*

Proof. Let $s = \sum_{i \geqslant 0} \mathrm{Sym}^i(x) t^i$ be the generating series of the polynomials Sym^i. Let us recall that from the recursive relation (37) it follows that

$$sxt = st^2 + s - 1,$$

or

$$s = 1/(t^2 - tx + 1). \tag{39}$$

We may think that, by (38), this is identified with the generating series for the v_i's as defined by (36), which is

$$(1+t)^{2n}(1-t^2), \tag{40}$$

but that is not quite right. The point is that (40) has non-trivial coefficients also at t^i with $i = n+2, \dots, 2n+2$ which (39) misses. The correct series equal to (40) is then

$$s(t) - t^{2n+2}s(t^{-1}) = (1 - t^{2n+4})s(t). \tag{41}$$

Thus, if a prime p divides the Braun–Douglas number for $\mathrm{Sp}(n)$, with representation level 1, then over $\mathbb{F}_p$,

$$t^{2n+4} - 1 = (t^2 - 2nt + 1)(1+t)^{2n+1}(t-1). \tag{42}$$

First let us note that, for $p = 2$, the right-hand side is $(1+t)^{2n+4}$, so (42) occurs if and only if $2n+4$ is a power of 2. To see that in this case, the Braun–Douglas number cannot be divisible by 4, consider the differences of the difference of dimension between the two sides of (37) for $i = 1$. Then the left-hand side is divisible by 4, while the right-hand side is $n(2n-1)$, which is not divisible by 4.

Now let us consider a prime $p \neq 2$. Let $p^j \| 2n+4$ and let $h = (2n+4)/p^j$. Then h is relatively prime to p, so $\overline{\mathbb{F}}_p$ actually has h different roots of $t^h - 1$. By looking at the right-hand side of (42), which has at most four different roots, we have $h \leqslant 4$. But h is divisible by 2, so $h = 2, 4$. If $j = 0$, then it is verified by direct computation that the Braun–Douglas number is 1. When $j > 0$, we see that the left-hand side contains at least p factors of $t-1$, while the right-hand side contains at most 3. So we would have to have $p = 3$, $j = 1$. Thus, $2n+4$ is equal to 6 or 12. Here 6 gives $n = 1$, which is excluded by assumption. The other case actually gives six copies of $t - 1$ on the left-hand side of (42), which cannot occur on the right-hand side. $\square$

We see therefore that the completion of the Verlinde algebra is a profound operation which can lose information. In the case $n = 2^\ell - 2$, it is also interesting to know the Atiyah–Hirzebruch filtration on the Verlinde algebra completion. First, we already know that the representation level 1 Verlinde algebra for $\mathrm{Sp}(n)$ with $n = 2^\ell - 1$ is

$$\mathbb{Z}/2[x]/(x^{2^n-1}). \tag{43}$$

Next, we construct polynomial generators $\gamma_1, \ldots, \gamma_n$ for the representation ring $R(\mathrm{Sp}(n))$ of $\mathrm{Sp}(n)$ such that γ_i is in the Atiyah–Hirzebruch filtration i. Obviously, for $i = 1$, we can just put

$$\gamma_1 = x - 2n. \tag{44}$$

Next, we put

$$\gamma_{i+1} = \Lambda^{i+1}(x - 2n + 2i) - \Lambda^{i-1}(x - 2n + 2i),\ i = 1, \ldots, n-1. \tag{45}$$

The element (45) is of filtration degree at least $i + 1$ because it vanishes when restricted to $\mathrm{Sym}(i)$ (where it is equal to $\mathrm{Sym}^{i+1}(x')$, where x' is the bottom level 1 representation of $\mathrm{Sp}(i)$).

THEOREM 7. *The associated graded ring of the representation* 1 *level Verlinde algebra of* $\mathrm{Sp}(2^r - 2)$ *with respect to the Atiyah–Hirzebruch filtration is isomorphic to*

$$\frac{\mathbb{Z}[\gamma_1, \ldots, \gamma_{2^r-2}]}{(2, \gamma_2+\gamma_1^2, \ldots, \gamma_{2^{r-1}-1}+\gamma_1^{2^{r-1}-1}, \gamma_{2^{r-1}+1}+\gamma_{2^{r-1}}\gamma_1, \ldots, \gamma_{2^r-2}+\gamma_{2^{r-1}}\gamma_1^{2^{r-1}-2}, \gamma_{2^{r-1}}\gamma_1^{2^{r-1}-1})}. \tag{46}$$

The element 2 *is represented by*

$$\gamma_{2^{r-1}} + \gamma_1^{2^{r-1}}. \tag{47}$$

Proof. Let the generating function of the $\Lambda^i(x)$'s be λ. Then (45) is equal to

$$\text{the coefficient at } t^{i+1} \text{ of } \lambda(x)(1 - t^2)/(1 + t)^{2n-2i}. \tag{48}$$

But we know

$$\lambda(x)(1 - t^2) = v = 1/(t^2 - tx + 1),$$

so (48) is equal to

$$\text{the coefficient at } t^{i+1} \text{ of } 1/((1 + t)^{2^{r+1}-4-2i}(t^2 - tx + 1)). \tag{49}$$

Let us first examine these polynomials mod 2. First, note that (49) is the coefficient at t^{i+1} of

$$(1 + t)^{4+2i}/(1 - tx + t^2).$$

Upon expanding the denominator in the variable $t(x - 2)$, we further get that this is the sum of coefficients at t^{i+1-j} of

$$(1 + t)^{2+2i-2j}(x - 2)^j, \tag{50}$$

which is $\binom{2+2i-2j}{1+i-j}$, which is odd if and only if $j = i + 1$. Let us also observe that

$$\begin{array}{l}\text{the coefficient at } t^{i+1-j} \text{ of (50) is 2 mod 4 if and only if}\\ i + 1 - j \text{ is a power of 2.}\end{array} \tag{51}$$

Thus, (51) will be the exact cases when the coefficient of (49) at x^j is 2 mod 4, with the exception of the case when $j = i$, in which the coefficient at x^i is 'anomalously' divisible by 4 when i is even.

Now let us examine the polynomials

$$\gamma_i + \gamma_1^i. \tag{52}$$

We just proved that the polynomials (52) are relations in the Verlinde algebra mod 2. This is what we got from the polynomial q_i obtained from γ_i by subtracting (49), substituting $\gamma_1 + 2^{r+1} - 4$ for x, and reducing mod 2. To proceed further, let us next look at the polynomial

$$q_{2^{r-1}}. \tag{53}$$

Since the coefficient at γ_1^0 of (49) is 2 mod 4 and all the coefficients at γ_1^j, with $0 < j < 2^{r-1}$, are even, we see that adding recursively multiples of $q_{2^{r-1}}\gamma_1^j$, with $0 < j < 2^{r-1}$, we obtain a relation in the Verlinde algebra of the form

$$2 + \gamma_1^{2^{r-1}} + \text{ higher filtration terms.} \tag{54}$$

This implies that 2 is a relation in the associated graded ring, and 2 is represented by (47). Next, processing in the same way q_i with $1 < i < 2^{r-1}$ (that is, adding recursively multiples of $q_{2^{r-1}}\gamma_1^j$, $0 < j < i$), we get (52) plus terms of higher filtration degree, which shows that (52) is a relation in the associated graded ring. For q_i with $2^{r-1} < i \leqslant 2^r - 2$, we use (51): the lowest coefficient of q_i at a power of γ_1 which is not divisible by 4 is at $\gamma_1^{i-2^{r-1}}$. Then add $\gamma_1^{i-2^{r-1}} q_{2^{r-1}}$ to make all coefficients at γ_1^j, $j < i$, divisible by 4. But 4 is represented in filtration at least 2^r, so we obtain the relation

$$\gamma_i + \gamma_1^{i-2^{r-1}} \gamma_1^{2^{r-1}} \tag{55}$$

in the associated graded ring as required.

Finally, the relation $\mathrm{Sym}^{2^r-1}(x)$ in the Verlinde algebra can be treated as $\gamma_{k+1} = \gamma_{2^r-1}$, giving the desired relation in this case also. We now see by a counting argument that the ring (46) is indeed additively the associated graded abelian group of the $\mathbb{Z}_2$-module $\mathbb{Z}_2^{2^r-1}$ with generators in degrees $0, \dots, 2^r - 2$, and 2 in degree 2^{r-1}, therefore our list of relations is complete. □

Example. Let us look at the lowest non-trivial case of Theorem 7, $r = 2$, so $n = 2$. Let us first compute the differentials of the twisted K-theory cohomology Serre spectral sequence of the fibration (28). We see from Theorem 7 that the only relation in the E_∞ term is

$$\gamma_1\gamma_2. \tag{56}$$

The vertical part of the spectral sequence is the twisted K-theory of $\mathrm{Sp}(2)$, which is the suspension by 3 (an odd number) of the $K_*/(2)$-exterior algebra on one generator ι in dimension 7. This is tensored with the horizontal part, which is $R = \mathbb{Z}[\gamma_1, \gamma_2]$. Thus, the E_2 term is (an odd suspension of)

$$\mathbb{Z}/2[\gamma_1, \gamma_2]\{1, \iota\}. \tag{57}$$

But now the R-submodule of (57) generated by 1 must disappear (to conform with the result of [**14**]), while the R-module generated by ι needs the single relation (56) (times ι). This means that we must have

$$d_{12}(1) = \iota\gamma_1\gamma_2. \tag{58}$$

(The right-hand side of (58) is actually still multiplied by an appropriate power of the Bott element.) The spectral sequence is a spectral sequence of modules over the horizontal part by Lemma 4.

Now let us consider the twisted K-theory Serre spectral sequence (representation level 1) of the fibration (27). We claim that the differential (58) is still the only one present. One sees in this case that the Borel words are $\mathrm{Sym}^{\ell+i}(\gamma_1)$, $i = 1, 2$ homogenized by multiplying by the appropriate powers of γ_2. Thus, by comparison with the case of (28), no differentials d_r, $r < 12$, are possible, and the ι-multiple of the E_{13} term is

$$\mathbb{Z}/2[\gamma_1, \gamma_2]/(\gamma_1\gamma_2, \gamma_1^i, \gamma_2^j), \tag{59}$$

where, for ℓ even, $j = (\ell/2) + 1$, $i = \ell + 1$, and for ℓ odd, $j = (\ell + 1)/2$, $i = \ell + 2$. Additionally, the multiple of 1 is the Poincaré dual of (59) times the top element of $H^*(G_{\mathrm{Sp}}(e \ll, 2))$. But by comparison with (28), the elements (59) are permanent cycles, and by Poincaré duality (see Section 2), so are the corresponding multiples of 1. Thus, the spectral sequence collapses to E_{13} in this case.

4. *More observations about the symplectic Verlinde algebras and their completions*

In this section, let $V(m, n)$ denote the $\mathrm{Sp}(n)$ Verlinde algebra of level m. If τ denotes the cohomological twisting associated to this level, then

$$K_\tau^i(LB\mathrm{Sp}(n)) \cong V(m, n)_I^\wedge \tag{60}$$

when $i \equiv n \bmod 2$, where I is the augmentation ideal of $\mathrm{RSp}(n)$. For $i \equiv (n + 1) \bmod 2$, the left-hand side of (60) is 0. The algebra $V(m, n)$ is a quotient of the representation ring $\mathrm{RSp}(n)$ by the ideal generated by the irreducible representations of level $m - n$. Explicitly, let x_1 be the defining representation of $\mathrm{Sp}(n)$ of dimension $2n$. Then for $n \geqslant k \geqslant 2$, there is a natural contraction

$$\Lambda^k(x_1) \longrightarrow \Lambda^{k-2}(x_2) \tag{61}$$

using the symplectic form. The map (61) is onto, and its kernel x_k is an irreducible representation of $\mathrm{Sp}(n)$. Here, $x_1, \ldots, x_n$ are precisely the irreducible representations of level 1.

A description of irreducible representations of $\mathrm{Sp}(n)$ of level q is given as polynomials in the variables $x_1, \ldots, x_n$ in [**15**, Proposition 24.24]. Let

$$A = (a_0, a_1, \ldots)$$

be the sequence

$$1, x_1, x_2, \ldots, x_n, 0, -x_n, \ldots, -x_1, 1.$$

The unspecified values of a_i are defined to be 0. Then define the *sequence A bent at i* as the sequence

$$a_i, a_{i+1} + a_{i-1}, a_{i+2} + a_{i-2}, \ldots.$$

Then irreducible representations of $\mathrm{Sp}(n)$ of level q correspond to Young diagrams with exactly q columns and at most n rows. Let the lengths of the columns be $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n$ (to recall, a Young diagram is precisely such sequence of numbers where $\mu_1 \leqslant n$). Then the corresponding irreducible representation is, in $\mathrm{RSp}(n)$, the determinant of the matrix whose ith row is given by the first q terms of the sequence A bent at $\mu_i - i + 1$.

In principle, the above description turns all algebraic questions about the $\mathrm{Sp}(n)$-Verlinde algebra into problems of commutative algebra. However, from this description, it is not always easy to see what is happening, and for general m, n, the algebra $V(m, n)$ and its completion are not completely understood. For example, there is a conjecture of Gepner [**16**] that the Verlinde algebra is a global complete intersection ring. Cummins [**9**] exhibited a 'level-rank' duality isomorphism

$$V(m, n) \cong V(m, m - n - 1). \tag{62}$$

The map (62) interchanges rows and columns in Young diagrams, so it sends x_i to the level i irreducible representation with $\mu_i = \ldots = \mu_1 = 1$.

Completion of the $\mathrm{Sp}(n)$-Verlinde algebra at the augmentation ideal of $\mathrm{RSp}(n)$ does not preserve the level-rank. In fact, we saw an example in Proposition 1 and Theorem 6. More generally, let $d(m,n)$ be the greatest common divisor of the dimensions of the irreducible representations of $\mathrm{Sp}(n)$ of level $m-n$.

PROPOSITION 8. *We have*

$$d(m,n) = \pm\mathrm{gcd}\left\{\sum_{j=-m}^{-1}\binom{2j+2(i-1)}{2(i-1)} \,|1 \leqslant i \leqslant n\right\}. \tag{63}$$

(*The right-hand side is the number calculated by Douglas* [**10**].)

Proof. Denote, for the moment, the number on the right-hand side of (63) by e. Then the twisted K-theory Serre spectral sequence, associated with the fibration

$$\mathrm{Sp}(n) \longrightarrow LB\mathrm{Sp}(n) \longrightarrow B\mathrm{Sp}(n),$$

along with the calculation [**10**] and the fact that the filtration on $K^0B\mathrm{Sp}(n)$ associated with the AHSS is the filtration by powers of the augmentation ideal, shows that

$$E^0_{I(R(\mathrm{Sp}(n)))}V(m,n) \text{ is a } \mathbb{Z}/e\text{-module,} \tag{64}$$

so in other words

$$d \mid e,$$

since the 0-slice of the filtration of the Verlinde algebra by $I(RS(n))$ is obtained by equating each x_i to its dimension, which gives $\mathbb{Z}/d$.

On the other hand, $K^0_\tau(G)$ is a module over $K^{G,0}_\tau(G)$ by restriction, which is a map of rings, while the augmentation ideal of $R(G)$ maps to 0, by considering the restriction $K^{G,0}(*) \to K^0(*)$, which is the augmentation. This implies

$$e \mid d. \qquad \square$$

As remarked above, the completion of $V(m,n)$ is additively a direct sum of a certain number of copies of $\mathbb{Z}_p$ over primes p which divide the number $d(m,n)$. Observe that by (63),

$$d(m,n+1) \mid d(m,n). \tag{65}$$

By Nakayama's lemma, the number of copies of $\mathbb{Z}_p$ is the same as the number of copies of $\mathbb{Z}/p$ in the completion of $V(m,n)/p$. If we denote by u_i the element x_i minus its dimension, then this is the same as taking the quotient of the power series ring $\mathbb{Z}/p[[u_1,\dots,u_n]]$ by the ideal J_{m-n} generated by representations of level $m-n$. Since this ideal has dimension 0, some powers $(u_i)^N$ must be in the ideal J_{m-n}. Since $V(m,n)/p$ itself however is a finite-dimensional $\mathbb{Z}/p$-vector space, N can be taken as its dimension, which is $\binom{m-1}{n}$. Using Maple, one can compute the Gröbner basis of the ideal generated by the level $m-n$ representations and $(u_i)^N$. This was done by Levin [**22**] in a number of examples.

Eventually, we detected a pattern, which allowed us to compute the completion of the symplectic Verlinde algebra in general, as well as the number $d(m,n)$. The results are contained in the following two theorems.

THEOREM 9. *For $n < m-1$, the completion of $V(m,n)$ at the augmentation ideal of* $\mathrm{RSp}(n)$ *is additively isomorphic to a sum of*

$$\binom{\delta(p,m)}{n} \tag{66}$$

copies of $\mathbb{Z}_p$ over all primes p.

Proof. When all symmetric polynomials of a finite collection of algebraic integers have positive p-valuation, so does each of them. Consider the maximal torus in $\mathrm{Sp}(n)$ which is given by embedding the product of n copies of a chosen maximal torus of $\mathrm{Sp}(1)$ via the standard embedding

$$\mathrm{Sp}(1) \times \ldots \times \mathrm{Sp}(1) \subset \mathrm{Sp}(n). \tag{67}$$

If we choose a generating weight t of $\mathrm{Sp}(1)$, then (67) gives generating weights $t_1, \ldots, t_n$ of $\mathrm{Sp}(n)$. Now the Grothendieck group of level 1 representations of $\mathrm{Sp}(n)$ is easily seen to have basis consisting of the elementary symmetric polynomials

$$\sigma_1, \ldots, \sigma_n$$

in

$$t_i + t_i^{-1} - 2, \quad i = 1, \ldots, n. \tag{68}$$

By the results of Freed–Hopkins–Teleman, the Verlinde algebra, when extended to a large enough ring of Witt vectors W, injects, with a finite cokernel, into a product of rings where in each individual factor, we quotient out by a relation setting σ_i equal to the ith symmetric polynomial in the numbers $N_1, \ldots, N_n$ obtained from (68) by setting

$$t_i = \zeta_{2m}^{j_i}, \quad 1 \leqslant j_1 < \ldots < j_n \leqslant m-1. \tag{69}$$

Since the σ_i's generate the augmentation ideal, the p-primary component of the completion of each of the factors is W if

$$\text{all the numbers } N_1, \ldots, N_n \text{ have positive } p\text{-valuation}, \tag{70}$$

and 0 otherwise

Now by the above remarks, (70) occurs if and only if all the numbers obtained by plugging in (69) into (68) have positive p-valuation. Now by recalling the argument in the proof of Proposition 1, this occurs if and only if each $\zeta_{2m}^{j_i}$ is a p^jth root of unity for some j. The number of such combinations is (66). $\square$

Note that in particular it follows that the p-component of the completion of $V(m,n)$ is isomorphic to the p-component of $V(m,n)$ if and only if $p=2$ and $m=2^r$ for some r. We also have a more explicit evaluation of $d(m,n)$.

THEOREM 10. *We have*

$$d(m,n) = n/\gcd(n,K), \tag{71}$$

where for each prime p, K is divisible by the largest power of p such that $\delta(p,K) < n$.

Proof. Despite the fact that Theorem 9 implies a part of Theorem 10 (namely, it detects when $p \mid d(m,n)$), we do not have a proof along the same lines at this point. Instead, we need to appeal to Proposition 8. Let

$$S(m,i) = \binom{2m-1}{1} + \binom{2m-3}{i} + \ldots + \binom{1}{i}. \tag{72}$$

Then by (63),

$$d(m,n)=\gcd\{S(m,0),S(m,2),\dots,S(m,2(n-1))\}. \tag{73}$$

Compute

$$\begin{aligned}\sum_{i\geqslant 0} S(m,i)x^i &= (1+x)^{2m-1}+(1+x)^{2m-3}+\ldots+(1+x)\\ &= (1+x)\frac{(1+x)^{2m}-1}{(1+x)^2-1}=\frac{(1+x)}{x(x+2)}((1+x)^{2m}-1).\end{aligned} \tag{74}$$

We see that the roots are

$$\begin{aligned}&-1,\\ &\zeta_{2m}^k-1,\quad k=1,\dots,2m-1,\ \ k\neq m.\end{aligned} \tag{75}$$

Now we distinguish two cases. The first case is $p=2$. Then consider the polynomial

$$P_\ell=\prod_{j=0}^{2^{\ell-1}-1}(x-(\zeta_{2^\ell}^{2j+1}-1)). \tag{76}$$

The coefficients of (76) (with the exception of the leading coefficient) are divisible by 2. We have $\deg(P_\ell)=2^{\ell-1}$. The polynomials P_ℓ have no common roots. Furthermore, the polynomial P_ℓ divides (74) precisely when $1\leqslant \ell\leqslant N$, where $2^N\|m$. Now $S(m,2(i-1))$ is a symmetric polynomial of degree $2m+1-2(i-1)$ in the roots (75). Decompose $S(m,2(i-1))$ as a polynomial in the coefficients of $P_1,\dots,P_N$. Observe that if

$$i<2^\ell, \tag{77}$$

then

$$2m+1-2(i-1)>(2m+1)-2^1-\ldots-2^s, \tag{78}$$

so each monomial of $S(m,2(i-1))$ considered as a polynomial in the roots of (74) contains roots of all the polynomials P_ℓ except, at most, $s-1$ of them. It then follows that $S(m,2(i-1))$ is divisible by $2^{n-\ell+1}$. On the other hand, if $i=2^s$, we see that (78) turns into an equality, so there exists precisely one monomial of $S(m,2(i-1))$ considered as a polynomial in the roots of (74) which does not contain roots of s of the polynomials P_ℓ (and contains all the other roots). Since $S(m,0)=m$, we conclude then that $2^{N-\ell}\|S(m,2(i-1))$. This is what we were aiming to prove.

Let us now consider the case $p>2$. In this case, we must consider the polynomials

$$Q_\ell=\prod\{(x-(\zeta_{p^\ell}^j-1))(x-(\zeta_{p^\ell}^{-1}-1))|j=1,\dots,p^{\ell-1},j\text{ not divisible by }p^{\ell-1}\}. \tag{79}$$

Then $\deg(Q_\ell)=(p-1)p^{\ell-1}$, the polynomials Q_ℓ have no common roots and $Q_1,\dots,Q_N$ divide (74) where $p^N\|m$. Observe that when

$$\frac{p^s-1}{2}<i, \tag{80}$$

then

$$2m+1-2(i-1)>(2m+1)-(p-1)-p(p-1)-\ldots p^{s-1}(p-1), \tag{81}$$

so again each monomial of $S(m,2(i-1))$, considered as a polynomial in the roots of (74), will contain roots of all the Q_ℓ's, except at most $s-1$ of them. Therefore, $S(m,2(i-1))$ is divisible by p^{N-s+1}.

On the other hand, when (80) turns into an equality, (81) turns into an equality, and therefore there is precisely one monomial in $S(m,2(i-1))$ considered as a polynomial in the roots of (74) which does not contain the roots of precisely s of the polynomials Q_ℓ (and contains all the other roots of (74)). Consequently, we can conclude that $p^{N-s}\|S(m,2(i-1))$, as needed. $\square$

Let us collect one more result, which we will come to use in the context of the next section. Recall that the Verlinde algebra is a Poincaré (that is, closed commutative Frobenius) ring where the augmentation ϵ is defined by $\epsilon(1) = 1$, and $\epsilon(a) = 0$ if a is a label different from 1. The only axiom of a Poincaré ring V (other than commutativity) is that the map

$$M : V \otimes V \longrightarrow \mathbb{Z} \tag{82}$$

defined by $\epsilon(ab)$ for variables $a, b \in V$ define an isomorphism

$$V \cong \mathrm{Hom}(V, \mathbb{Z}). \tag{83}$$

One then has an inverse of (83), which can be interpreted as a map

$$N : \mathbb{Z} \longrightarrow V \otimes V. \tag{84}$$

The composition (84) with the triple product is the '1-loop translation operator', which we denote by T.

THEOREM 11. *For every $m \geqslant n + 2$,*

$$\det(T) \neq 0. \tag{85}$$

In $V(m, 1)$, we have

$$\det(T) = (-2)^{m-1} m^{m-3}. \tag{86}$$

Proof. By the Verlinde conjecture (which is known to be true for $\mathrm{Sp}(n)$), the $\mathbb{C}$-Poincaré algebra $V(m, n) \otimes \mathbb{C}$ is isomorphic to a product of 1-dimensional algebras, which are then automatically Poincaré algebras, which implies that $T \otimes \mathbb{C}$ is always an invertible matrix, which implies (85). To prove (86), we recall the formula (3). Let ζ be the primitive $2m$th root of unity. In the $m - 1$ direct factors of $V(m, 1)$, we will then have

$$x = \zeta^j + \zeta^{-j}, \quad j = 1, \ldots, m - 1. \tag{87}$$

The labels are

$$1, \mathrm{Sym}^1(x), \ldots, \mathrm{Sym}^{m-2}(x), \tag{88}$$

and they are self-contragredient, so by the restriction T_i of T to the ith summand (87), we have

$$T_i = \sum_{j=0}^{m-2} \mathrm{Sym}^j(\zeta^i + \zeta^{-i})^2. \tag{89}$$

Using the well-known formula

$$\mathrm{Sym}^j(x) = \frac{(x + \sqrt{x^2 - 1})^{j+1} - (x - \sqrt{x^2 - 1})^{j+1}}{2\sqrt{x^2 - 1}}, \tag{90}$$

we get

$$\mathrm{Sym}^j(\zeta^i + \zeta^{-i}) = \frac{\zeta^{i(j+1)} - \zeta^{-i(j+1)}}{\zeta^i - \zeta^{-i}}, \tag{91}$$

which gives

$$T_i = \frac{-2m}{(\zeta^i - \zeta^{-i})^2}. \tag{92}$$

The determinant of T is then the product of the numbers (88) over $i = 1, \ldots, m - 1$, which is (86). □

It is worth noting that computer calculations using Maple suggest the conjecture

$$\det(T) = 2^{(m-1)\binom{m-3}{n-2}} m^{(m-3)\binom{m-3}{n-2}}. \tag{93}$$

We shall prove here a weaker statement.

Theorem 12. *When for a prime p, $P \mid d(m,n)$, $m > 3$, then $p \mid \det(T)$ in $V(m,n)$.*

Proof. Let ζ be a primitive $2m$th root of unity. Then by [**14**], $V(m,n) \otimes \overline{\mathbb{Q}}$ splits as a product of Poincaré algebras V_I, where $I = (1 \leqslant i_1 < \ldots < i_n < m)$ and V_I is the quotient of $V(m,n) \otimes \overline{Q}$ by the ideal generated by $x_i - \alpha_i$, $i = 1, \ldots, n$, where x_i are the level 1 irreducible representations, and α_i are the numbers obtained by expressing $_i$ as a polynomial in the standard weights t_i, and evaluating

$$t_j = \zeta^{i_j}.$$

(Here t_j correspond to choosing a maximal torus T in $\mathrm{Sp}(1)$ and then $T^n \subset \mathrm{Sp}(1)^n \subset \mathrm{Sp}(n)$, where the latter is the standard embedding.)

Now V_I, as a $\overline{Q}$-algebra, is isomorphic to $\overline{Q}$. To specify its structure as a Poincaré algebra, one must evaluate

$$e_I = \epsilon(1), \tag{94}$$

where ϵ is the augmentation. On the other hand, in a 1-dimensional Poincaré algebra, it is easy to check that

$$T = 1/\epsilon(1). \tag{95}$$

So

$$T_I = 1/e_I, \tag{96}$$

where T_I is the T-operator on V_I. Now let us change from $\overline{Q}$ to $\overline{Q_p}$. Then one also has the formula

$$T = \sum_{k \in K} x_k x_k^*,$$

where x_k, $k \in K$, are the irreducible representations of representation level at most $m - n$. This shows that, if we denote by v the p-valuation, then

$$v(T_I) \geqslant 0,$$

so, by (96), we have

$$v(e_I) \leqslant 0. \tag{97}$$

We need to show that when

$$p \mid d(m,n), \tag{98}$$

the inequality (97) is sharp for at least one I. Assume therefore (98), and that we have

$$v(e_I) = 0 \tag{99}$$

for all I. Recall now that the augmentation ϵ satisfies

$$\begin{aligned} \epsilon(1) &= 1, \\ \epsilon(x_k) &= 0, \quad x_k \neq 1, \ k \in K. \end{aligned} \tag{100}$$

Denote by $x_{k,I}$ the number obtained from k_k by plugging in ζ^{i_j} for t_j, $j = 1, \ldots, n$ (using the Weyl character formula for x_k). Then $B = (x_k, I)$ is a square matrix, and (99) and (100) signify that the equation

$$Bx = (1, 0, \ldots, 0)^{\mathrm{T}} \tag{101}$$

(the right-hand side has 1 in the kth position, where $x_k = 1$, and 0 elsewhere) has a solution in $\overline{Z_p}$ (the solution being $(e_I)^{\mathrm{T}}$). But now note that the 'twisted augmentation' ϵ_k given by

$$\begin{aligned} \epsilon_k(x_k) &= 1, \\ \epsilon_k(x_\ell) &= 0, \quad \ell \neq k \end{aligned} \tag{102}$$

is given simply by

$$\epsilon_k(x) = \epsilon(x \cdot x_k^*). \tag{103}$$

Therefore, *all* equations

$$Bx = (0, \ldots, 0, 1, 0, \ldots, 0)^{\mathrm{T}}, \tag{104}$$

where 1 is in the kth position for any $k \in K$, have a solution in $\mathbb{Z}_p$. Therefore, B is an invertible matrix, and

$$v \det B = 0. \tag{105}$$

But now $m > 3$, so $1 + 2m/p < m$ (for $p > 2$). Now consider $I = (i_1 < i_2 < \ldots < i_n)$, where

$$\begin{aligned} &i_1 = 1, \\ &i_j \neq 1 + \frac{2m}{p} \quad \text{for any } j = 1, \ldots, n, \end{aligned}$$

and let J be obtained from I by replacing i_1 with $1 + 2m/p$. Since

$$v(\zeta - \zeta^{1+2m/p}) > 0,$$

the Ith and Jth column of B are congruent modulo $v > 0$, and hence

$$v(\det(B)) > 0,$$

which is a contradiction. For $p = 2$, replace $2m/p$ by $m/2 = 2m/4$. □

5. *String topology operations in twisted K-theory: the product*

In [**17**], Godin defines a family of string topology operations on $H_*(LM)$, parametrized by the homology $H_*(\Gamma_{g,n})$ of mapping class groups. This extended Chas–Sullivan's proof in [**4**] that $H_*(LM)$ is a Batalin–Vilkovisky algebra. It seems likely that analogues of these operations are present in the twisted K-homology $K_*^\tau(LM)$, for suitable choices of τ. In this section we focus on the most basic operation, the loop product, and compute it for $K_*^\tau(L\mathbb{H}P^n)$.

5.1. *Basic recollections*

Let X be a topological space and $\tau \in H^3(X)$ be a twisting. Recall that τ defines a bundle $E_\tau = (E_i)_{i\in\mathbb{Z}}$ of spectra over X with fibre the K-theory spectrum. The twisted K-homology is

$$K_n^\tau(X) = \pi_n(E_\tau/X) = \varinjlim_i \pi_{i+n}(E_i/X),$$

where X is regarded as a subspace of E_i via a section.

Functoriality is not as straightforward as for untwisted theories. For any map $f : Y \to X$, there is a pullback bundle $E_{f^*(\tau)}$ over Y, equipped with a map to E_τ covering f. Consequently there is an induced map

$$f_* : K_n^{f^*(\tau)}(Y) \longrightarrow K_n^\tau(X).$$

Cross products also require some care. If $\sigma \in H^3(Z)$, then consider the element $(\tau, \sigma) := p_1^*(\tau) + p_2^*(\sigma) \in H^3(X \times Z)$ where p_i are projections onto X and Z. This defines a bundle of K-theory spectra $E_{(\tau,\sigma)}$ over $X \times Z$. The smash product $E_\tau \wedge E_\sigma$ also defines a bundle of spectra over $X \times Z$; in this case the fibre is $K \wedge K$. Multiplication in this ring spectrum gives a map $E_\tau \wedge E_\sigma \to E_{(\tau,\sigma)}$, which in turn defines an exterior cross product

$$\times : K_n^\tau(X) \otimes K_m^\sigma(Z) \longrightarrow K_{m+n}^{(\tau,\sigma)}(X \times Z).$$

Now assume X is a homotopy associative, homotopy unital H-space with multiplication $\mu : X \times X \to X$. Composing the external cross product with μ_* gives the following.

Lemma 13. *If τ is primitive, that is, $\mu^*(\tau) = (\tau, \tau)$, then μ makes $K_*^\tau(X)$ into a ring.*

5.2. *The loop product*

We shall need the following, adapted from, for example, [**14**, Section 3.6]:

Proposition 14. *Let $f : Y \to X$ be an embedding of finite codimension, with K-orientable normal bundle N of dimension d. There is an umkehr map*

$$f^! : K_n^\tau(X) \longrightarrow K_{n-d}^{f^*(\tau)}(Y)$$

for any $\tau \in H^3(X)$.

As usual, this is obtained from a Pontrjagin–Thom collapse and Thom isomorphism (which is just a shift desuspension since we are taking N to be K-orientable).

Putting this together with the work of Chas and Sullivan [**4**] and Cohen and Jones [**6**] gives us a loop product in twisted K-theory; namely, let M^d be a closed smooth manifold of dimension d, and write

$$LM = \mathrm{Map}(S^1, M)$$

for the space of piecewise smooth maps $S^1 \to M$.

For simplicity, assume that M is 3-connected; as a consequence, it is spin and hence K-orientable. Let $\tau' \in H^4(M)$; then connectivity ensures that there is a well-defined transgressed class $\tau \in H^3(\Omega M)$. Through the Serre spectral sequence τ gives rise to a well-defined class (represented by $1 \otimes \tau$) which we shall also denote by $\tau \in H^3(LM)$. Connectivity implies, further, that $\tau \in H^3(\Omega M)$ is primitive.

Theorem 15. *Assume that M is 3-connected. Then $K_{*+d}^\tau(LM)$ is a unital, associative ring.*

Proof. As usual, we consider the following commutative diagram in which the lower left square is cartesian:

$$\begin{tikzcd}
\Omega M \times \Omega M \arrow[d] & \Omega M \times \Omega M \arrow[l, "="'] \arrow[r, "\mu"] \arrow[d] & \Omega M \arrow[d] \\
LM \times LM \arrow[d, "\mathrm{ev}\times\mathrm{ev}"'] & LM \times_M LM \arrow[l, "\tilde{\Delta}"'] \arrow[r, "\mathrm{concat}"] \arrow[d, "\mathrm{ev}_\infty"'] & LM \arrow[d, "\mathrm{ev}"] \\
M \times M & M \arrow[l, "\Delta"'] \arrow[r, "="] & M.
\end{tikzcd}$$

Primitivity of τ means that $\mu^*(\tau) = (\tau, \tau)$ on ΩM. Therefore $\mathrm{concat}^*(\tau) = \tilde{\Delta}^*(\tau, \tau)$ on LM.

The inclusion $\tilde{\Delta}$ is of finite codimension with normal bundle $N \cong \mathrm{ev}_\infty^*(TM)$, which is K-orientable, by assumption. Therefore, we may form the composite $m := \mathrm{concat}_* \circ \tilde{\Delta}^! \circ \times$:

$$\begin{tikzcd} K_n^\tau(LM) \otimes K_m^\tau(LM) \arrow[r, "m"] \arrow[d, "\times"'] & K_{n+m-d}^\tau(LM) \\ K_{n+m}^{(\tau,\tau)}(LM \times LM) \arrow[r, "\tilde{\Delta}^!"] & K_{n+m-d}^{\tilde{\Delta}^*(\tau,\tau)}(LM \times_M LM). \arrow[u, "\mathrm{concat}_*"'] \end{tikzcd}$$

Then the arguments given in [6] in homology show that m is an associative and unital product. $\square$

This multiplication intertwines with the cup product in the untwisted K-theory of M in the following way: $\mathrm{ev} : LM \to M$ has a right inverse $c : M \to LM$; $c(p)$ is the constant loop at p. Then c induces a map $c_* : K_*^{c^*(\tau)}(M) \to K_*^\tau(LM)$. However, since M is 3-connected, $c^*(\tau) = 0$, this is simply

$$c_* : K_*(M) \longrightarrow K_*^\tau(LM).$$

If we give the $K_*(M)$ a ring structure via intersection theory (Poincaré dual to the cup product), this is clearly a ring homomorphism.

PROPOSITION 16. *The twisted string K-theory, $K_*^\tau(LM)$ is a module over $K^*(M)$ via the map c_*.*

Note that unless $\tau = 0$, there is no class $\tau' \in H^3(M)$ with the property that $\mathrm{ev}^*(\tau') = \tau$. Consequently, ev does not induce a map $\mathrm{ev}_* : K_*^\tau(LM) \to K_*(M)$ which splits the target off from the source (as is the case in the untwisted setting). Indeed, we shall see in examples that the source is often torsion, while the target is often not.

5.3. *A Cohen–Jones–Yan type spectral sequence*

In [7], Cohen–Jones–Yan constructed a spectral sequence converging to $H_{*+d}(LM)$ as an algebra. Combining their arguments with the AHSS for twisted K-theory gives the following.

THEOREM 17. *Let M be a 3-connected closed d-manifold, and choose $\tau' \in H^4(M)$ with associated transgressed twisting $\tau \in H^3(LM)$. There is a left half-page spectral sequence $\{E_{p,q}^r : -d \leqslant p \leqslant 0\}$ satisfying the following.*

(i) *The differentials $d^r : E_{p,q}^r \to E_{p-r,q+r-1}^r$ are derivations.*

(ii) *The spectral sequence converges to $K_{*+d}^\tau(LM)$ as an algebra (where we use the loop product on ΩM).*

(iii) *Its E^2 term is given by*

$$E_{p,q}^2 := H^{-p}(M, K_q^\tau(\Omega M)).$$

5.4. *An example: $\mathbb{H}P^\ell$*

Since $\mathbb{H}P^\ell$ is even-dimensional, there is no degree shift in the ring $K_0^\tau(L\mathbb{H}P^\ell)$. We let $\tau' \in H^4(\mathbb{H}P^\ell) \cong \mathbb{Z}$ correspond to $m \in \mathbb{Z}$.

Theorem 18. *For any $m \neq 0$, $K_*^\tau(L\mathbb{H}P^\ell)$ is concentrated in even degrees, and $K_0^\tau(L\mathbb{H}P^\ell)$, equipped with the loop product, is isomorphic to*

$$\prod_{p|m} \mathbb{Z}_p[t,y]/(y^{\ell+1}, \sigma^{m-1}(y) - (\ell+1)y^\ell t).$$

We will prove this using the spectral sequence from the previous section. First we need the following lemma.

Lemma 19. *There is a ring isomorphism*

$$K_*^\tau(\Omega\mathbb{H}P^\ell) \cong (K_*/m)[t],$$

where $|t| = 4\ell + 2$.

Proof. We use the twisted K-theory Serre spectral sequence associated with the fibration

$$\Omega S^{4\ell+3} \longrightarrow \Omega\mathbb{H}P^\ell \longrightarrow \mathrm{Sp}(1). \tag{106}$$

The E_2 term is

$$K^*\Omega S^{4\ell+3} \otimes \Lambda[x_3]. \tag{107}$$

The first factor is vertical and the second is horizontal. The torsion on the vertical factor disappears because of connectivity. Further, using the AHSS,

$$K^*\Omega S^{4\ell+3} = K^* \otimes H^*(\Omega S^{4\ell+3}) = K^* \otimes \mathbb{Z}\{t_{q(4\ell+2)} | q = 0, 1, 2 \ldots\}^\wedge. \tag{108}$$

(The $?^\wedge$ on the right-hand side indicates that in K-cohomology, we have the product-completion, that is, a product of copies of $\mathbb{Z}$ rather than a direct sum. Actually, (108) is the product-completion of a divided power algebra. No extensions are possible, since there is no torsion. Regarding the differentials in (107), we know that d_3 is multiplication by the twisting class, which (by our choice) is mx_3. Thus, the $E_4 = E_\infty$ term is a suspension by x_3 of

$$K^*\Omega S^{4\ell+3}/(m). \tag{109}$$

Regarding extensions, first note that the element represented by x_3 really is m-torsion, since our spectral sequence is a spectral sequence of modules over the AHSS for the twisted K-theory of $\mathrm{Sp}(1)$. Next, (107) is always a spectral sequence of modules over its untwisted analogue. However, there $d_3 = 0$ by the same argument, and further the $t_{q(4\ell+2)}$'s are permanent cycles by filtration considerations (there are no elements in filtration degrees greater than 3). Thus, the element represented by $t_{q(4\ell+2)}x_3$ is a product of the elements represented by $t_{q(4\ell+2)}$ and x_3 in the untwisted and twisted spectral sequences, respectively, and hence is m-torsion.

Additively, the result in twisted K-homology follows by the universal coefficient theorem. Multiplicatively, it follows from the fact that $H_*(\Omega S^{4\ell+3}) = \mathbb{Z}[t]$. □

Proof of Theorem 18. Lemma 19 implies that the spectral sequence for $K_*^\tau(L\mathbb{H}P^\ell)$ has the E^2 term given by

$$E^2_{*,*} = K_*[t,y]/(m, y^{\ell+1}), \tag{110}$$

where y has filtration degree -4. The spectral sequence collapses at E^2 since it is concentrated in even degrees. We therefore know that the ring structure is given by

$$K_0^\tau(L\mathbb{H}P^\ell) = \mathbb{Z}[y,t]/(y^{\ell+1}, \sigma^{m-1}(y) - tp(y,t)) \tag{111}$$

for some polynomial $p(y,t)$, since, by Proposition 27, the map $\tilde{h} : L\mathbb{H}P^\ell \to Y(\ell, 1)$ induces a ring map in K_*^τ. Now recall (say, [25]) the ordinary homology Serre spectral sequence of the fibration

$$\Omega\mathbb{H}P^{\ell} \longrightarrow L\mathbb{H}P^{\ell} \longrightarrow \mathbb{H}P^{\ell}.$$

The only differentials are

$$d(t^j y_\ell u) = (\ell+1)t^{j+1},$$

where y_i denotes the generator of $H_{4i}(\mathbb{H}P^\ell)$. Therefore, we can conclude that $\Sigma^{-4\ell}H_*(L\mathbb{H}P^\ell,\mathbb{Z})$, with its string topology multiplication, is given by

$$\mathbb{Z}[y,t,v]/(y^{\ell+1},v^2,vy^\ell,(\ell+1)ty^\ell), \tag{112}$$

where $\dim(y)=-4$, $\dim(v)=-1$, and $\dim(t)=4\ell+2$. Now applying the twisted K-homology AHSS to (112), we get the twisting differentials

$$d_3(vy^i)=my^{i+1}. \tag{113}$$

We see that the odd-dimensional subgroup of E^4 is generated by

$$q_i=\frac{\ell+1}{\gcd(\ell+1,m)}t^iy^{\ell-1},\quad i>0. \tag{114}$$

Let us consider the case of $i=1$ in (114). The lesser filtration degree part of the spectral sequence is

$$\mathbb{Z}/m\{y,y^2,\dots,y^\ell\}\oplus\mathbb{Z}\{1\}. \tag{115}$$

By mapping into the twisted K-homology AHSS for $Y(\ell,1)$, we know that no element of

$$\mathbb{Z}/m\{1,y,y^2,\dots,y^\ell\} \tag{116}$$

can be the target of a differential. We conclude therefore that

$$d_5(q_1)=N.1,\quad N\neq 0 \tag{117}$$

for some number N. Additionally, recalling the elements (116) are not targets of differentials, we see that we must have

$$m\mid N. \tag{118}$$

In fact, by the multiplicative structure, the differential (117) remains valid when we multiply by a power of the permanent cycle t, so we see that the AHSS collapses to E^6. We conclude that we must have

$$(\ell+1)ty^\ell=q(y) \tag{119}$$

for some polynomial $q(y)$. This means that the polynomial $(\ell+1)ty^\ell-q(y)$ must belong to the ideal generated by $y^{\ell+1}$ and $\sigma^{m-1}(y)+tp(y,t)$. By reducing $p(y,t)$ to degree at most ℓ in the y variable, this clearly implies

$$p(y,t)\mid(\ell+1)y^\ell. \tag{120}$$

On the other hand, by our computation of the AHSS, the only divisors of $(\ell+1)y^\ell$ that can have lower filtration degree are multiples of

$$\gcd(m,\ell+1)y^\ell. \tag{121}$$

But now note that any such polynomial is congruent to $\mathbb{Z}/m$-unit times $(\ell+1)y^\ell$ modulo the relation my^ℓ, which will be valid once we know that $p(y,t)$ is divisible by (121). Finally, we may change basis by multiplying t by a $\mathbb{Z}/m$-unit, making the unit equal to 1. □

Comment. One now sees that $N=m^2/\gcd(m,\ell+1)$.

We can now determine the precise additive structure of $K_0^\tau(L\mathbb{H}P^\ell)_{(p)}$. This is essentially a standard exercise in abelian extensions of p-groups. Despite some simplifications coming from the ring structure, there are many eventualities, and we find it easiest to use a geometric pattern to represent the answer.

Let us, first, represent in this way the additive structure of $K_0^\tau(Y(\ell,1)_{(p)})$, as calculated in Section 1. Let d be $(p-1)/2$ if $p>2$ and 1 if $p=2$. Imagine a table T with k rows indexed by $0 \leqslant i < k$ and $\ell+1$ columns indexed by $0 \leqslant j \leqslant \ell$. The field (i,j) represents the element $p^i y^j$ in the AHSS. The extensions are determined by paths (a step in the path represents multiplication by p). The paths look as follows: we start from a field $(0,j)$, where

$$\frac{p^a-1}{p-1}d \leqslant j < \frac{p^{a+1}-1}{p-1}d, \quad 0 \leqslant a < \delta(p,m). \tag{122}$$

For future reference, we shall call j *critical* if equality arises for j in (122). Now in each path, proceed one field up all the way to row $k-a-1$:

$$(0,j),\ (1,j),\ \ldots\ (k-a-1,j).$$

In the same row, now, proceed by

$$\frac{p^{a+1}-1}{p-1}d$$

rows to the right,

$$(k-a-1,j), \left(k-a-1, j+\frac{p^{a+1}-1}{p-1}d\right), \ldots$$

until we run out of columns in the table T. The lengths of the paths are now the exponents of the p-powers which are orders of the summands of $K_0^\tau(Y(\ell,1)_{(p)})$. Let us introduce some terminology: first, introduce an ordering on the fields of T: $(i',j') < (i,j)$ if (i',j') precedes (i,j) on a path (the paths are disjoint so this creates no ambiguity). Next, by the *divisibility* of a field (i,j) we shall mean the number of fields less than (i,j) in our order (clearly, this represents the exponent of the power of p by which the element the field represents is divisible).

To describe $K_0^\tau(L\mathbb{H}P^\ell)_{(p)}$, imagine copies T_n of the table T, where $n=0,1,2,\ldots$. We label the (i,j)-field in T_n by (n,i,j), and it will represent the element $p^i a^j t^n$ in the associated graded abelian group given by Theorem 3. We start with the disjoint union of the tables T_n with their own paths. However, now the paths (and the corresponding order) will be corrected as follows: inductively in n, some fields in T_n (all in column ℓ) will be deleted from paths in T_n and appended to paths in T_{n-1}. The procedure is as follows. Let $p^c \| r$, $r=\gcd(m,\ell+1)$. For a critical generator $(n,0,j)$ (recall (122)), let α_j be the number of fields in its path in the highest row of T_n which the path reaches, which are not taken over by paths in T_{n-1}, minus 1, plus c. Then append to the path of $(n,0,j)$, in increasing row order, all fields $(n+1, c+\alpha+d, \ell)$ whose divisibility in T_{n+1} is at most

$$\alpha+d+k-a, \quad d=0,1,\ldots,$$

(recall (122) for the definition of a), and which were not already appended to the paths of critical generators $(n,0,j')$ with $j'<j$.

Note that only one set of corrections arises for each n, so it is easy to determine the length of the corrected paths (which are the orders of the generators of direct summands of $K_0^\tau(L\mathbb{H}P^\ell)_{(p)}$. However, note that a number of scenarios can occur, and we know of no simple formula describing the possible results in one step.

Comment. It is interesting to note that, by the additive extensions we calculated, $K_*^\tau(L\mathbb{H}P^\ell)$ with its string product cannot be a module over $K_*^\tau(\Omega\mathbb{H}P^\ell)$ with its loop product structure. This exhibits the subtlety of the structures involved here. However, as in [**4**, Proposition 3.4], there is a ring map $K_*^\tau(LM) \to K_{*\text{-}\dim M}^\tau(\Omega M)$ given by intersection with the subspace of based loops. Hence $K_*^\tau(\Omega\mathbb{H}P^\ell)$ is a module over $K_*^\tau(L\mathbb{H}P^\ell)$.

6. *The loop coproduct*

The twisted string K-theory ring $K^\tau_{*-d}(LM)$ also admits a coalgebraic structure by reversing the role of concat and $\tilde{\Delta}$ (and the associated Pontrjagin–Thom maps) in the definition of the loop product.

6.1. *The coproduct*

Note that one may regard concat as an embedding of a finite-codimension submanifold, just as for $\tilde{\Delta}$. Specifically, there is a cartesian diagram

$$\begin{tikzcd} LM \times_M LM \arrow[r, "\text{concat}"] \arrow[d, "\mathrm{ev}_\infty"'] & LM \arrow[d, "\mathrm{ev}_{1,-1}"] \\ M \arrow[r, "\Delta"] & M \times M \end{tikzcd}$$

where $\mathrm{ev}_{1,-1}$ evaluates a loop both at the basepoint $1 \in S^1$, as well as the midpoint $-1 \in S^1$. The pullback over the diagonal is the subspace of loops that agree at ± 1. By reparametrization, this space may be identified with $LM \times_M LM$, and the inclusion with the concatenation of loops.

Consequently concat admits a shriek map in twisted K-theory, as well. Again, using the fact that $\text{concat}^*(\tau) = \tilde{\Delta}^*(\tau,\tau)$, we may consider the composite

$$K^\tau_{n+d}(LM) \xrightarrow{\text{concat}^!} K_n^{\tilde{\Delta}^*(\tau,\tau)}(LM \times_M LM) \xrightarrow{\tilde{\Delta}_*} K_n^{(\tau,\tau)}(LM \times LM).$$

For many purposes this map suffices. However, to properly define a coproduct, we must assume that the exterior cross product map $\times$ is an isomorphism; the preferred way to do this (using the Künneth theorem) is to take our coefficients for K-theory to be in a field $\mathbb{F}$. Define

$$\nu := \times^{-1} \circ \tilde{\Delta}_* \circ \text{concat}^! : K^\tau_{*+d}(LM;\mathbb{F}) \longrightarrow K^\tau_*(LM;\mathbb{F}) \otimes_{K_*} K^\tau_*(LM;\mathbb{F}).$$

THEOREM 20. *The map ν defines the structure of a coassociative coalgebra on $K^\tau_{*-d}(LM;\mathbb{F})$.*

We do not expect ν to be counital. Were that the case, $K^\tau_*(LM;\mathbb{F})$ would be equipped with a non-degenerate trace, and thus finite-dimensional. But, as we have seen in Theorem 2, this is not generally the case.

6.2. *The IHX relation*

The composite $\nu \circ m$ of the loop product and coproduct satisfies the same relations as in a Frobenius algebra.

PROPOSITION 21. *If we write left or right multiplication of $K^\tau_*(LM)$ on $K^\tau_*(LM) \otimes K^\tau_*(LM)$ by $\cdot$, then*

$$\nu(xy) = x \cdot \nu(y) = \nu(x) \cdot y.$$

Proof. Consider the diagram:

$$\begin{tikzcd}
LM \times LM & LM \times (LM \times_M LM) \arrow[l, "1\times\mathrm{concat}"'] \arrow[r, "1\times\tilde\Delta"] & LM \times LM \times LM \\
LM \times_M LM \arrow[u, "\tilde\Delta"] \arrow[d, "\mathrm{concat}"'] & LM \times_M LM \times_M LM \arrow[l, "1\times\mathrm{concat}"'] \arrow[r, "1\times\tilde\Delta"] \arrow[u, "\tilde\Delta\times 1"'] \arrow[d, "\mathrm{concat}\times 1"'] & (LM \times_M LM) \times LM \arrow[u, "\tilde\Delta\times 1"'] \arrow[d, "\mathrm{concat}\times 1"'] \\
LM & LM \times_M LM \arrow[l, "\mathrm{concat}"'] \arrow[r, "\Delta"] & LM \times LM.
\end{tikzcd}$$

Going around the top and right of the diagram, replacing wrong way maps by the associated umkehr map (that is, 'push–pull') gives $x \cdot \nu(y)$. Push–pull along the left and bottom gives $\nu(xy)$. All of the squares except the lower left are cartesian, and the lower left is homotopy cartesian. Consequently the two push–pull sequences are equal. A similar diagram proves that $\nu(x \cdot y) = \nu(x) \cdot y$. □

6.3. *Stabilization over genus*

The '1-loop translation operator' of Theorem 11 is of considerable interest. In the context of $K_*^\tau(LM)$, T is the other composite of the coproduct and product:

$$T = m \circ \nu : K_*^\tau(LM) \longrightarrow K_{*-2d}^\tau(LM).$$

In the full field-theoretic language of string topology (which, to our knowledge, has not been constructed in the twisted K-theory setting), this is the operation induced by the class of a point in $B\Gamma_{1,1+1}$, the moduli of Riemann surfaces of genus 1 with 1 incoming and 1 outgoing boundary.

Note that, since we have assumed that M is K-orientable, the tangent bundle gives an element $TM \in K^0(M)$.

DEFINITION 22. Let $E \in K^0(M)$ be the K-theoretic Euler class of TM:

$$E := \sum_{k=0}^{d} (-1)^k \Lambda^k(TM).$$

We recall that $K_*^\tau(LM)$ is a module over $K^0(M)$ via cap products along constant loops (Proposition 16).

THEOREM 23. *In $K_*^\tau(LM)$, T is given by cap product with the square of E:*

$$T = E^2.$$

Proof. This is essentially a consequence of Atiyah–Singer's computation of shriek maps in K-theory [3]. Recall that if $i : Y \to X$ is an embedding of finite codimension with K-oriented normal bundle N, then the composite

$$i^* i_! : K^*(Y) \longrightarrow K^*(Y)$$

is given by multiplication by $\sum_k (-1)^k \Lambda^k(N) \in K^0(Y)$. Further, if N can be written $N \cong i^* N'$, where N' is a K-oriented bundle on X, then the reverse composite

$$i_! i^* : K^*(X) \longrightarrow K^*(X)$$

is multiplication by $\sum_k (-1)^k \Lambda^k(N') \in K^0(X)$. The same phenomena are true in twisted K-theory via the module structure over untwisted K-theory.

We now apply this to concat and $\tilde{\Delta}$. The normal bundles to each of these embeddings are isomorphic to

$$\mathrm{ev}_\infty^*(TM) \longrightarrow LM \times_M LM,$$

where ev_∞ evaluates at the common point of the two loops. In both cases, $\mathrm{ev}_\infty^*(TM)$ is a pullback:

$$\mathrm{ev}_\infty^*(TM) \cong \mathrm{concat}^*(\mathrm{ev}^*(TM)) \quad \text{and} \quad \mathrm{ev}_\infty^*(TM) \cong \tilde{\Delta}^*(\mathrm{ev}^*(TM) \times 0).$$

Write

$$E' = \sum_k (-1)^k \Lambda^k(\mathrm{ev}_\infty^*(TM)) \in K^0(LM \times_M LM)$$

and note that $\mathrm{concat}^*(\mathrm{ev}^* E) = E'$. Then, for $x \in K_\tau^*(LM)$, the dual map to T is

$$\begin{aligned} T^*(x) &= \nu^*(m^*(x)) \\ &= \mathrm{concat}_! \tilde{\Delta}^*(\tilde{\Delta}_! \mathrm{concat}^*(x)) \\ &= \mathrm{concat}_!(E' \cdot \mathrm{concat}^*(x)) \\ &= \mathrm{concat}_!(\mathrm{concat}^*(\mathrm{ev}^* E \cdot x)) \\ &= (\mathrm{ev}^* E)^2 \cdot x. \end{aligned}$$

Translating this into K-homology turns the cup product into a cap product. □

Unfortunately, however, we have the following.

Lemma 24. *We always have $E^2 = 0$.*

Proof. The square, E^2 is the Euler class of the Whitney sum of two copies of the tangent bundle of M. However, in any bundle of dimension greater than that of M, the 0-section can be moved off itself by general position, and hence the Euler class is 0. (Clearly, the same argument shows that the product of the Euler class of the tangent bundle with any Euler class of any positive-dimensional vector bundle is 0 in any generalized cohomology theory with respect to which M is oriented.) □

Therefore, we have the following corollary.

Corollary 25. *We have $T = 0$.*

6.4. *The coproduct on $K_*^\tau(L\mathbb{H}P^\ell)$*

Despite these 'negative results', we can use the method of Theorem 23 to obtain non-trivial information about the coproduct. Let us consider the string coproduct for $M = \mathbb{H}P^\ell$ with coefficients in $\mathbb{Z}/p$. Concretely, we use the fact that the composition

$$K_0^\tau(LM \times_M LM, \mathbb{Z}/p) \xrightarrow{\mathrm{concat}_*} K_0^\tau(LM, \mathbb{Z}/p) \xrightarrow{\mathrm{concat}^!} K_0^\tau(LM \times_M LM, \mathbb{Z}/p) \tag{123}$$

is given by cap product with the pullback of the Euler class of M. Using the notation of Theorem 18, $1, t \in K_0^\tau(LM, \mathbb{Z}/p)$ are in the image of $\mathrm{concat}_\sharp$, so we have

$$\mathrm{concat}^!(1) = (\ell + 1) y^\ell, \tag{124}$$

$$\mathrm{concat}^!(t) = (\ell+1)y^\ell t \tag{125}$$

(since the Euler class is $(\ell+1)y^\ell$; recall also that the cup product in $K^*(M)$ is Poincaré dual to the string product on constant loops). From (124), using Poincaré duality, we immediately conclude that

$$\nu(1) = (\ell+1)y^\ell \otimes y^\ell. \tag{126}$$

Regarding (125), recall that the right-hand side is equal to

$$\sigma^{m-1}(y).$$

Consider the following condition:

$$c_i = \mathrm{coeff}_{y^\ell}(\sigma^{m-1}(y)) \text{ is not divisible by } p \text{ for } i=\ell, \text{ and is divisible by } p \text{ for all } i<\ell. \tag{127}$$

If the condition (127) fails, then either $p \mid (\sigma^{m-1}(y), y^{\ell+1})$ in which case the relation on $K_0^\tau(LM, \mathbb{Z}/p)$ reads $(\ell+1)y^\ell t = 0$, and

$$\nu(t) = 0, \tag{128}$$

or there exists an $i < \ell$ such that p does not divide c_i. However, then $y^{\ell-i}\sigma^{m-1}(y) = y^\ell \bmod p$, so $y^\ell = 0$, and hence (128) also occurs.

If condition (127) is satisfied, then the relation of Theorem 18 with coefficients in $\mathbb{Z}/p$ reads

$$(\ell+1)y^\ell t = c_\ell y^\ell,$$

so the product formula implies

$$\nu(t) = c_\ell = y^\ell \times y^\ell. \tag{129}$$

Proposition 26. *The condition (127) is equivalent to*

$$\ell = \delta(p,m). \tag{130}$$

Thus, if (130) holds, we have (129), else we have (128).

Proof. The first statement follows from our calculation of $K_0^\tau(Y(\ell,1))$ in Section 1. The second statement is proved in the above discussion. □

7. *Completions and comparison with the Gruher–Salvatore pro-spectra*

All of the manifolds that we have considered as examples, symplectic Grassmannians, come in families associated to particular compact Lie groups. Indeed, our computation of $K_*^\tau(L\mathbb{H}P^\ell)$ depends very much upon our computation of the completion of the Verlinde algebra for Sp(1). In this section, we make that connection more explicit.

There is a technical difficulty involved: while the manifolds we consider come in a sequence, for example,

$$\mathbb{H}P^1 \longrightarrow \mathbb{H}P^2 \longrightarrow \mathbb{H}P^3 \longrightarrow \cdots,$$

it is not the case that the induced maps on free loop spaces preserve string topology operations (since intersection theory is contravariant, not covariant). Nonetheless, one would like to study string topology operations on this system.

7.1. *Adjoint bundles*

Gruher–Salvatore achieve this goal using an interesting construction that approximates the sequence

$$L\mathbb{H}P^1 \longrightarrow L\mathbb{H}P^2 \longrightarrow L\mathbb{H}P^3 \longrightarrow \cdots,$$

and, furthermore, preserves their modification of the string topology product.

To be more specific, we let G denote a compact Lie group, and choose a model for BG. We take as given an infinite sequence of even-dimensional, closed, K-oriented manifolds $B_\ell G$, equipped with a commutative diagram of embeddings

$$\begin{tikzcd} \cdots \arrow[r, "i_{\ell-1}"] & B_\ell G \arrow[r, "i_\ell"] \arrow[d, "j_\ell"'] & B_{\ell+1}G \arrow[r, "i_{\ell+1}"] \arrow[dl, "j_{\ell+1}"] & \cdots \\ & BG & & \end{tikzcd}$$

with the property that the connectivity of j_ℓ increases with ℓ. Consequently the induced map

$$j : \varinjlim B_\ell G \longrightarrow BG$$

is a homotopy equivalence. Let $E_\ell G = j_\ell^*(EG)$ be the pullback of the universal principal G-bundle over BG, with quotient $B_\ell G = E_\ell G/G$. Lastly, define the *adjoint bundle*

$$\mathrm{Ad}(E_\ell G) := G \times_G E_\ell G,$$

where G acts on itself by conjugation; this is a G-bundle over $B_\ell G$.

For instance, if $G = U(1)$, we may take

$$E_\ell G = S^{2\ell+1} \subseteq \mathbb{C}^{\ell+1} \setminus \{0\} \quad \text{and} \quad B_\ell G = \mathbb{C}P^\ell.$$

In this case, since G is abelian, $\mathrm{Ad}(E_\ell G) = \mathbb{C}P^\ell \times U(1)$.

Alternatively, if $G = \mathrm{Sp}(n)$, then we may proceed via the manifolds described in Sections 1 and 2; that is, we may take $B_\ell(\mathrm{Sp}(n))$ to be the symplectic Grassmannian

$$B_\ell(\mathrm{Sp}(n)) = G_{\mathrm{Sp}}(\ell, n)$$

of n-dimensional $\mathbb{H}$-submodules of $\mathbb{H}^{\ell+n}$. Then $E_\ell(\mathrm{Sp}(n))$ is the Stiefel-manifold $W(\ell, n)$ of symplectic ℓ-frames in $\mathbb{H}^{\ell+n}$, and

$$\mathrm{Ad}(E_\ell(\mathrm{Sp}(n))) = W(\ell, n) \times_{\mathrm{Sp}(n)} \mathrm{Sp}(n) = Y(\ell, n)$$

which is a non-trivial bundle over $G_{\mathrm{Sp}}(\ell, n)$.

We are interested in $\mathrm{Ad}(E_\ell G)$ because it provides an approximation to $LB_\ell G$ and LBG. Consider the pair of fibrations

$$\begin{tikzcd} \Omega(B_\ell G) \arrow[r] \arrow[d, "h"] & LB_\ell G \arrow[r] \arrow[d, "\tilde{h}"] & B_\ell G \arrow[d, "="] \\ G \arrow[r] & \mathrm{Ad}(E_\ell G) \arrow[r] & B_\ell G, \end{tikzcd}$$

where the vertical map h is the holonomy of a loop; in homotopy-theoretic language, it is given by the first map in the fibration sequence

$$\Omega(B_\ell G) \xrightarrow{h} G \longrightarrow E_\ell G \longrightarrow B_\ell G.$$

As ℓ tends to infinity in this sequence, h tends to a homotopy equivalence, since $E_\ell G$ becomes increasingly connected. Thus, as ℓ tends to infinity, $\tilde{h} : LB_\ell G \to \mathrm{Ad}(E_\ell G)$ tends to a homotopy equivalence.

7.2. *The product on $K_*^\tau(\mathrm{Ad}(E_\ell G))$*

Gruher–Salvatore define a ring multiplication on $h_*(\mathrm{Ad}(E_\ell G))$ for any cohomology theory with respect to which the vertical tangent bundle of $\mathrm{Ad}(EG) \to BG$ is oriented. The multiplication is an intermediary between the string topology product on $LB_\ell G$ and the fusion product on ${}^GK_\tau^*(G)$; it mixes intersection theory on $B_\ell G$ with multiplication in G. Gruher extends these results in [**18**] to show that in fact $h_*(\mathrm{Ad}(E_\ell G))$ is a Frobenius algebra over h_* when h_* is a graded field.

One may introduce twistings to this story. Assume now that G is simply connected, so that every $\tau \in H^3(G)$ is primitive. Denote also by τ the twistings in $H^3(LB_\ell G)$ and $H^3(\mathrm{Ad}(E_\ell G))$ associated to τ by the Serre spectral sequence. Then Gruher shows that $K_*^\tau(\mathrm{Ad}(E_\ell G))$ admits the structure of a ring via the same multiplication as in [**19**].

Specifically, one may define the product on $K_*^\tau(\mathrm{Ad}(E_\ell G))$ via a push–pull construction. There is a commutative diagram of fibrations

$$\begin{tikzcd}
G \times G \arrow[d] & G \times G \arrow[l, "="'] \arrow[r, "\mu"] \arrow[d] & G \arrow[d] \\
\mathrm{Ad}(E_\ell G) \times \mathrm{Ad}(E_\ell G) \arrow[d] & \mathrm{Ad}(E_\ell G) \times_{B_\ell G} \mathrm{Ad}(E_\ell G) \arrow[l, "\tilde{\Delta}"'] \arrow[r, "\tilde{\mu}"] \arrow[d] & \mathrm{Ad}(E_\ell G) \arrow[d] \\
B_\ell G \times B_\ell G & B_\ell G \arrow[l, "\Delta"'] \arrow[r, "="] & B_\ell G.
\end{tikzcd}$$

Here, $\tilde{\mu}$ is fibrewise multiplication in G. Then the product is defined as $m := \tilde{\mu}_* \circ \tilde{\Delta}^! \circ \times$. The proof that twistings behave appropriately is the same as in the construction of the loop product in Section 5.

Proposition 27. *The map*

$$\tilde{h}_* : K_*^\tau(LB_\ell G) \longrightarrow K_*^\tau(\mathrm{Ad}(E_\ell G))$$

is a ring homomorphism.

Proof. In the diagram

$$\begin{tikzcd}
LB_\ell G \times LB_\ell G \arrow[d, "\tilde{h}\times\tilde{h}"'] & LB_\ell G \times_{B_\ell G} LB_\ell G \arrow[l, "\tilde{\Delta}"'] \arrow[r, "\mathrm{concat}"] \arrow[d, "h'"'] & LB_\ell G \arrow[d, "\tilde{h}"'] \\
\mathrm{Ad}(E_\ell G) \times \mathrm{Ad}(E_\ell G) & \mathrm{Ad}(E_\ell G) \times_{B_\ell G} \mathrm{Ad}(E_\ell G) \arrow[l, "\tilde{\Delta}"'] \arrow[r, "\tilde{\mu}"] & \mathrm{Ad}(E_\ell G)
\end{tikzcd}$$

the right square homotopy commutes, since h may be taken to be an H-map. The left square is in fact cartesian, so

$$\begin{aligned}
\tilde{h}_*(x \cdot y) &= \tilde{h}_*\mathrm{concat}_*\tilde{\Delta}^!(x \times y) \\
&= \tilde{\mu}_* h'_* \tilde{\Delta}^!(x \times y) \\
&= \tilde{\mu}_* \tilde{\Delta}^!(\tilde{h}_*(x) \times \tilde{h}_*(y)) \\
&= \tilde{h}_*(x) \cdot \tilde{h}_*(y).
\end{aligned}$$

□

This result, combined with the high connectivity of $\tilde{h}$ (for large ℓ) can be taken as an indication that $K_*^\tau(\mathrm{Ad}(E_\ell G))$ is an increasingly good approximation for the string topology multiplication on $K_*^\tau(LB_l G)$.

7.3. *Limits*

The inclusions $E_\ell G \to E_{\ell+1}G$ are G-equivariant, so induce the inclusions $\mathrm{Ad}(E_\ell G) \to \mathrm{Ad}(E_{\ell+1}G)$. A Pontrjagin–Thom collapse for this embedding (or equivalently, Poincaré duality in twisted K-theory) defines a map

$$K_*^\tau(\mathrm{Ad}(E_{\ell+1}G)) \longrightarrow K_*^\tau(\mathrm{Ad}(E_\ell G)),$$

which does not shift degrees, since $\mathrm{codim}(B_\ell G \subseteq B_{\ell+1}G)$ is even. It is a consequence of [**19**] that this is a ring homomorphism. So this gives rise to an inverse system of rings

$$\cdots \longrightarrow K_*^\tau(\mathrm{Ad}(E_{\ell+1}G)) \longrightarrow K_*^\tau(\mathrm{Ad}(E_\ell G)) \longrightarrow \cdots \longrightarrow K_*^\tau(\mathrm{Ad}(E_0 G)).$$

THEOREM 28. *Let $h(G)$ be the dual Coxeter number of G. There is a ring isomorphism*

$$\varprojlim K_*^\tau(\mathrm{Ad}(E_\ell G)) \cong V(\tau - h(G), G)_I^\wedge,$$

where the left-hand side is the inverse limit of the string topology ring structures on $K_*^\tau(\mathrm{Ad}(E_\ell G))$, and the right-hand side is the completion of the Verlinde algebra (*with fusion product*) *at the augmentation ideal.*

Proof. The inverse system of Pontrjagin–Thom collapse maps

$$\cdots \longrightarrow \mathrm{Ad}(E_{\ell+1}G)^{-TB_{\ell+1}G} \longrightarrow \mathrm{Ad}(E_\ell G)^{-TB_\ell G} \longrightarrow \cdots \longrightarrow \mathrm{Ad}(E_0 G)^{-TB_0 G}$$

is Spanier–Whitehead dual to the direct system of inclusions

$$\cdots \supseteq \mathrm{Ad}(E_{\ell+1}G) \supseteq \mathrm{Ad}(E_\ell G) \supseteq \cdots \supseteq \mathrm{Ad}(E_0 G)$$

and so there is an isomorphism

$$\varprojlim K_*^\tau(\mathrm{Ad}(E_\ell G)) \cong \varprojlim K_\tau^{-*}(\mathrm{Ad}(E_\ell G)). \tag{131}$$

The main result of [**18**] is that for untwisted homology theories, this duality throws the string topology product onto the 'fusion product'. The same is true in the twisted setting: as we have seen, the multiplication on $K_*^\tau(\mathrm{Ad}(E_\ell G))$ is given by the formula $m := \tilde{\mu}_* \circ \tilde{\Delta}^! \circ \times$. This is evidently Poincaré dual to the product

$$p : K_\tau^*(\mathrm{Ad}(E_\ell G)) \otimes K_\tau^*(\mathrm{Ad}(E_\ell G)) \longrightarrow K_\tau^*(\mathrm{Ad}(E_\ell G))$$

defined as the composite $p = \tilde{\mu}_! \circ \tilde{\Delta}^* \circ \times$. Therefore (131) is a ring isomorphism.

Furthermore, since $\varinjlim \mathrm{Ad}(E_\ell G) = \mathrm{Ad}(EG) = G \times_G EG$, a $\lim^1$ argument implies that

$$\varprojlim K_\tau^*(\mathrm{Ad}(E_\ell G)) \cong K_\tau^*(\mathrm{Ad}(EG)) \cong {}^G K_\tau^*(G \times EG).$$

The last is the G-equivariant twisted K-theory of $G \times EG$. By the twisted version of the Atiyah–Segal completion theorem [**11**, **21**], this is isomorphic to ${}^G K_\tau^*(G)_I^\wedge$. Combining these results gives a ring isomorphism

$$\varprojlim K_*^\tau(\mathrm{Ad}(E_\ell G)) \cong {}^G K_\tau^*(G)_I^\wedge,$$

where the multiplication on the right-hand side of the isomorphism is given by the G-equivariant transfer $\mu_!$ to the principal G-bundle $\mu : G \times G \to G$. The main theorem in [**14**] then gives the desired isomorphism. That this isomorphism preserves the ring structure is immediate, as we note that the authors in [**14**] show that the fusion product on $V(\tau - h(G), G)$ is carried to the product on ${}^G K_\tau^*(G)$ defined as the transfer to μ. □

We note that a completion is a form of inverse limit; namely,

$$V(\tau - h, G)_I^\wedge = \varprojlim V(\tau - h, G)/I^l.$$

We are thus led to wonder if the isomorphism of Theorem 28 is in fact realized on a geometric level.

Conjecture 29. There is an increasing function $N : \mathbb{N} \to \mathbb{N}$ and a collection of isomorphisms

$$f_\ell : K_*^\tau(\mathrm{Ad}(E_\ell G)) \cong V(\tau - h, G)/I^{N(\ell)}$$

which are coherent across the inverse system, that is, they induce the isomorphism of Theorem 28 upon passage to the limit.

This is true in the case of $G = \mathrm{Sp}(1)$ (where $N(\ell) = \ell + 1$), as evidenced by Theorem 18.

7.4. *The coproduct*

Gruher also defines a coproduct on the twisted K-theory $K_*^\tau(\mathrm{Ad}(E_\ell G); \mathbb{F})$ with field coefficients; it is given by

$$\nu = \times^{-1} \circ \tilde{\Delta}_* \circ \tilde{\mu}^! : K_*^\tau(\mathrm{Ad}(E_\ell G); \mathbb{F}) \longrightarrow K_*^\tau(\mathrm{Ad}(E_\ell G); \mathbb{F}) \otimes K_*^\tau(\mathrm{Ad}(E_\ell G); \mathbb{F}).$$

For untwisted homology theories, this map is the Poincaré dual to the product, and in fact forms part of a Frobenius algebra structure. It is unclear whether the same is true in the twisted setting, since the constant map $\mathrm{Ad}(E_\ell G) \to pt$ does not induce a map in twisted K-homology.

Nonetheless, the Pontrjagin–Thom map

$$K_*^\tau(\mathrm{Ad}(E_{\ell+1} G)) \longrightarrow K_*^\tau(\mathrm{Ad}(E_\ell G))$$

preserves the coproduct; consequently the inverse limit

$$\varprojlim K_*^\tau(\mathrm{Ad}(E_\ell G)) \cong V(\tau - h(G), G)_I^\wedge$$

admits a coproduct induced by ν. It is not at all obvious whether this is related to the coproduct derived from the Frobenius algebra structure on the Verlinde algebra.

However, it is *not* true that the map

$$\tilde{h}_* : K_*^\tau(LB_\ell G; \mathbb{F}) \longrightarrow K_*^\tau(\mathrm{Ad}(E_\ell G); \mathbb{F})$$

is a homomorphism of coalgebras. One can try to mimic the proof of Proposition 27, but the argument fails, since the right square is not in fact cartesian. Indeed, $\mathrm{concat}_!$ is given by intersection theory, whereas $\mu_!$ is given by a G-transfer.

8. *Concluding remarks*

We have examined several different field theories in this paper. The Verlinde algebra $V(m, G)$ is a Poincaré algebra or, equivalently, a topological quantum field theory (TQFT). We have constructed a product and coproduct on the twisted string K-theory of a manifold $K_*^\tau(LM)$. By virtue of the IHX relation (Proposition 21), these combine in such a way as to give $K_*^\tau(LM)$ the structure of a 'TQFT without trace' or 'positive boundary TQFT', as was done in the homological setting by [**5**]. The adjoint bundle K-theories $K_*^\tau(\mathrm{Ad}(E_\ell G))$ serve as an imperfect bridge between these, preserving products, but not necessarily coproducts.

Despite this connection, when it comes to higher-genus operations, $V(m, G)$ and $K_*^\tau(LM)$ display markedly different behaviour, as evidenced by the vanishing of T in $K_*^\tau(LM)$, and its rational invertibility in $V(m, \mathrm{Sp}(n))$. It is natural to ask for the reasons behind these differences. To approach this question, consider yet another form of field theory: Gromov–Witten theory.

We would like to think of the Verlinde algebra as a twisted K-theoretic analogue of Gromov–Witten theory for the stack $[*/G]$. This point of view is supported by papers such as [**13**], where further field-theoretic operations on $V(m, G)$ are examined.

String topology and Gromov–Witten theory share some ideas in their construction, at least on a schematic level. They both involve a push–pull diagram of the form

$$(LX)^m \longleftarrow \mathrm{Map}(\Sigma, X) \longrightarrow (LX)^n, \tag{132}$$

where Σ is a surface with $m + n$ boundary components, the maps are restrictions along boundaries, and the various function spaces are to be interpreted in the right categories.

The shriek map in Gromov–Witten theory applies a type of intersection theory, using the deep fact of the existence of a virtual fundamental class on the compactification of the moduli of maps $\Sigma \to X$. In contrast, the one in string topology applies a Becker–Gottlieb type transfer using the fairly straightforward fact that the maps in (132) are fibrations with compact fibre when $X = BG$. Said in another way, in string topology, the fibres of the maps in (132) are compact, whereas in Gromov–Witten theory, the spaces themselves are compact (to the eyes of cohomology, at least).

As hinted in the introduction, in this paper, we were interested in K-theory information. In conformal field theory, an example of such information is the Verlinde algebra, or more precisely modular functor of a CFT. According to modern string theory, the physical aspects of strings, such as branes or even physical partition function, are also based on K-theory. If we look at homology with characteristic 0 coefficients instead of K-theory, topological quantum field theories should come out of the 'state space' of the conformal field theory itself. In effect, when $N = (2, 2)$ supersymmetry is present, we can construct such models, namely the A-model and the B-model. This also fits into the Gromov–Witten picture: $N = (2, 2)$ supersymmetric conformal field theories are expected to arise not from ordinary manifolds, but from Calabi–Yau varieties. While a rigorous direct construction of such models is not known, Fan, Jarvis, and Ruan [**12**] constructed the topological A-model, along with TQFT structure, and coupling to compactified gravity, in the related case of the Landau–Ginzburg orbifold, via applying Gromov–Witten theory to the Witten equation. One can then ask if one can somehow extend these methods to obtain K-theory rather than characteristic 0 homology information.

Appendix. *Some foundations*

It is not the purpose of the present paper to discuss in detail the foundations of twisted K-theory. Many of the results needed here can be read off from the approach of [**1**, **2**]. More systemic approaches, needed for some of the more complicated assertions, use the setup of parametric spectra, which has been developed, for example, in [**23**] or [**20**] (the latter approach is simpler but requires some corrections). The main point is to note that the required foundations exist, even though they are somewhat scattered throughout the literature. Let us recapitulate here some key points.

Let us work in the category of parametric K-modules over a space X and let us denote this category by K-mod$/X$. Then the projection $p : X \to *$ (actually, the point can be replaced by any space) has a pullback map

$$p^* : K\text{-mod}/* \longrightarrow K\text{-mod}_X,$$

which has a left adjoint $p_\sharp$ and a right adjoint p_*.

The category K-mod$/X$ has an internal ('fibrewise') smash product $\wedge_{K/X}$ and function spectrum $F_{K/X}$. Twistings τ are objects of K-mod$/X$ which are invertible under $\wedge_{K/X}$. There are the usual 'exponential' adjunctions.

Now let us consider the universal coefficient theorem. It is better to deal with K-modules than with coefficients. The τ-twisted K-homology module is $p_\sharp\tau$ and the τ-twisted K-cohomology module is $p_*\tau$. We have

$$\begin{aligned} F_{K/*}(p_\sharp\tau, K) &= p_*F_{K/*}(\tau, p^*K) = p_*F_{K/*}(\tau, 1) \\ &= p_*F_{K/*}(1, \tau^{-1}) = p_*\tau^{-1}. \end{aligned} \tag{A.1}$$

So this is the usual universal coefficient theorem; the contribution of the twisting is that it gets inverted. Note, however, that the complex conjugation automorphism of K-theory reverses the sign of twisting, so K-(co)-homology groups with opposite twistings are isomorphic.

Let us now discuss Poincaré duality: when X is a closed (finite-dimensional) manifold, its K-dualizing object ω is K fibrewise smashed with its stable normal bundle (considered as a parametric spectrum over X). Up to suspension by the dimension d of X, ω is a twisting. When $\omega = 1[-d]$, X is called K-orientable (so when, for example, $H^3(X, Z) = 0$, X is K-orientable). Poincaré duality states that for a parametric K-module over X,

$$p_*? = p_\sharp(\omega\wedge_{K/X}?). \tag{A.2}$$

So, indeed, when the manifold is orientable, its K_τ homology and cohomology are the same up to dimension shift.

Using these foundations, usual results on K-theory extend immediately to the twisted context. For example, there is a twisted Serre (and hence Atiyah–Hirzebruch) spectral sequence converging to twisted K-homology or cohomology. Also, the above foundations can be extended to the equivariant case, in which case there is a completion theorem, asserting that, for G compact Lie and a finite G-CW complex X, the non-equivariant twisted K-cohomology of $X \times_G EG$ is isomorphic to the completion of the equivariant twisted K-theory of X, completed at the augmentation ideal of $R(G) = K^*_G(*)$. For a detailed proof of this result, we refer the reader to [**11**, **21**].

Let us make one more remark on notation. We shall be using both the induced map and the transfer for a closed inclusion of manifolds $f : X \to Y$. Another form of duality then states that

$$f_! \simeq f_*. \tag{A.3}$$

Here $f_!$ is obtained from $f_\sharp$ by composing with the smash product with the sphere bundle which is the fibrewise 1-point compactification of the normal bundle of X in Y (alternatively, take the corresponding invertible parametrized K-module, and smash over K_X). Our interest in this is that we need to consider the induced map and transfer map of a smooth inclusion f. First of all, for a twisting τ on Y, if we denote by $[?, ?]$ homotopy classes of maps of parametrized K-modules, 1_Y the trivial K-module on Y, and by τ a twisting on Y, clearly we have a map

$$f^* : [1_Y, \tau] \longrightarrow [1_X, f^*\tau], \tag{A.4}$$

as $1_X = f^*1_Y$. Assuming now (say) that the normal bundle of X in Y is K-orientable of real codimension k, we can produce a map in the other direction

$$f_! : [1_X, f^*\tau] \longrightarrow [1_Y, \tau[k]] \tag{A.5}$$

by taking a map $1_X = f^*1_Y \to f^*\tau$, taking the adjoint $1_Y \to f_*f^*1_Y$, using (A.3), and composing with the counit of the adjunction $?^*$, $?_\sharp$.

We see now that the notation (A.4) and (A.5) is not really justified in terms of parametrized K-modules: in effect, it is 'one level below in terms of 2-category theory', and fits more with the base change functors applied to vector bundles which represent classes in the cohomology groups involved in (A.4) and (A.5). This makes selecting notation for the corresponding maps in cohomology precarious. As a compromise between possibly contradicting allusions, we use

f_* for the induced map in homology and $f^!$ for the transfer. Thus, f_* (in twisted K-homology) is induced by the adjunction counit

$$f_\sharp f^* \tau \longrightarrow \tau,$$

$f^!$ (in homology) is induced by the adjunction unit

$$\tau \longrightarrow f_* f^* \tau,$$

together with (A.3).

We should note that in this paper, we also use infinite extensions of these duality results, allowed by the work of Cohen and Klein [**8**].

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