EDITED 4Θ-EMBEDDINGS OF JACOBIANS

GREG W. ANDERSON

Abstract. By the Lefschetz embedding theorem a principally polarized g-dimensional abelian variety is embedded into projective space by the linear system of 4^g half-characteristic theta functions. Suppose we *edit* this linear system by dropping all the theta functions vanishing at the origin to order greater than parity requires. We prove that for Jacobians the edited 4Θ linear system still defines an embedding into projective space. Moreover, we prove that the projective models of Jacobians arising from the elementary algebraic construction of Jacobians recently given by the author are (after passage to linear hulls) copies of the edited 4Θ model. We obtain our results by aptly combining the quartic and determinantal identities satisfied by the Riemann theta function. We take the somewhat nonstandard tack of working in the framework of Weil's old book on Kähler varieties in order to avoid having to make extremely complicated calculations.

1. Introduction

The point of departure for this paper is the elementary algebraic construction of Jacobians given in [Anderson 2002]. We begin by reviewing that construction. For brevity's sake we reformulate the construction in terms of line bundles rather than divisors. Let X be a nonsingular projective algebraic curve of genus $g > 0$. Although the main work of this paper takes place over the complex numbers, for the moment we take as ground field any algebraically closed field. Fix an integer $n \geq g+2$. For $i = 0, \ldots, n+1$, let $f \mapsto f^{(i)}$ denote the operation of pull-back via the i^{th} projection $X^{\{0,\ldots,n+1\}} \to X$. Fix a line bundle $\mathcal{E}$ on X of degree $n+g-1$. Given any line bundle $\mathcal{T}$ on X of degree 0, let u (resp. v) be a row vector of length n with entries forming a basis of $H^0(X, \mathcal{T}^{-1} \otimes \mathcal{E})$ (resp., $H^0(X, \mathcal{T} \otimes \mathcal{E})$) over the ground field, and let

2000 *Mathematics Subject Classification.* Primary: 14H40, 14H42.

$\mathrm{abel}(\mathcal{T})$ be the n by n matrix with entries

$$\mathrm{abel}(\mathcal{T})_{ij} := \left|\begin{array}{c}\widehat{v^{(0)}}\\ \vdots \\ \widehat{v^{(i)}}\\ \vdots\end{array}\right| \cdot \left|\begin{array}{c}\vdots \\ \widehat{u^{(i)}}\\ \vdots \\ \widehat{u^{(n+1)}}\end{array}\right| \cdot \left|\begin{array}{c}\vdots \\ \widehat{v^{(j)}}\\ \vdots \\ \widehat{v^{(n+1)}}\end{array}\right| \cdot \left|\begin{array}{c}\widehat{u^{(0)}}\\ \vdots \\ \widehat{u^{(j)}}\\ \vdots\end{array}\right|,$$

where the leftmost determinant is that obtained by stacking the row vectors $v^{(i)}$ to form an $n+2$ by n matrix with rows numbered from 0 to $n+1$, then striking row 0 and row i to obtain a square matrix, and finally taking the determinant; the other determinants are analogously formed. Up to a nonzero scalar multiple the matrix $\mathrm{abel}(\mathcal{T})$ is independent of the choice of bases u and v and moreover depends only on the isomorphism class of the line bundle $\mathcal{T}$. It is easy to see that $\mathrm{abel}(\mathcal{T})$ does not vanish identically. The construction $\mathcal{T} \mapsto \mathrm{abel}(\mathcal{T})$ maps the set of isomorphism classes of degree zero line bundles on X to the projective space of lines in the space of n by n matrices with entry in i^{th} row and j^{th} column drawn from the space

$$H^0\left(X^{\{0,\dots,n+1\}}, \frac{\bigotimes_{\ell=0}^{n+1}\left(\mathcal{E}^{(\ell)}\right)^{\otimes 4}}{\left(\mathcal{E}^{(0)}\right)^{\otimes 2}\otimes\left(\mathcal{E}^{(i)}\right)^{\otimes 2}\otimes\left(\mathcal{E}^{(j)}\right)^{\otimes 2}\otimes\left(\mathcal{E}^{(n+1)}\right)^{\otimes 2}}\right).$$

In [Anderson 2002]:

- the map $\mathcal{T} \mapsto \mathrm{abel}(\mathcal{T})$ is shown to be injective,
- an explicit set of homogeneous equations cutting out the image of the map $\mathcal{T} \mapsto \mathrm{abel}(\mathcal{T})$ is exhibited, and
- the image of the map $\mathcal{T} \mapsto \mathrm{abel}(\mathcal{T})$ is explicitly equipped with an algebraic group law commuting with the tensor product of degree zero line bundles.

Thus the Jacobian of X is constructed in elementary algebraic fashion.

The question motivating this paper is the following:

> What linear system of effective divisors of the Jacobian arises from the projective embedding $\mathcal{T} \mapsto \mathrm{abel}(\mathcal{T})$?

We attack the question by the methods of complex algebraic geometry and obtain a complete answer in that setting. The question remains open in positive characteristic.

The answer we finally obtain is surprisingly simple. In the classical-style language of [Fay], [Mumford Tata I] and [Mumford Tata II] the answer takes the following form. The curve X is now a compact Riemann surface. Let τ be the period matrix of X. Let M be the set of column vectors of length $2g$ with entries in the set $\{0, 1/2\}$. We write such column vectors in block form $\left[\begin{array}{c} a \\ b \end{array}\right]$ where a and b are both of length g. Let $M_{\leq 1}$ be the subset of M consisting of those $\left[\begin{array}{c} a \\ b \end{array}\right]$ such that the corresponding half-characteristic classical theta function

$$\theta \left[\begin{array}{c} a \\ b \end{array}\right](w, \tau) := \sum_{\ell \in \mathbb{Z}^g} \exp\left(\pi i(\ell + a)^T \tau(\ell + a) + 2\pi i(\ell + a)^T (w + b)\right)$$

vanishes at $w = 0$ to order not exceeding 1, i. e., to order not greater than that dictated by parity considerations. Then—independent of the choice of n and the line bundle $\mathcal{E}$—it turns out that the linear system we are looking for is the *edited* 4Θ linear system

$$\left\{ \theta \left[\begin{array}{c} a \\ b \end{array}\right](2w, \tau) \middle| \left[\begin{array}{c} a \\ b \end{array}\right] \in M_{\leq 1} \right\}.$$

(Here and throughout the paper we abuse language by identifying linear systems of effective divisors with linearly independent sets of sections of line bundles in the obvious way.) Of course, for this answer to make sense, the edited 4Θ linear system has to embed the Jacobian into projective space, i. e., the "edited version" of the Lefschetz embedding theorem has to hold. The latter we prove in this paper (see Theorem 3.1.3 below) by complex analytic methods independent of—but largely parallel to—the methods of [Anderson 2002]. The theorem has some content since, for example, $M_{\leq 1} \neq M$ for all hyperelliptic curves of large genus. We conclude the paper by explaining in detail how to factor the map $\mathcal{T} \mapsto \mathrm{abel}(\mathcal{T})$ through the edited 4Θ-embedding, thus fully answering the motivating question over the complex numbers.

The main technical problem faced in this paper is that of aptly expressing and combining the quartic and determinantal identities satisfied by the Riemann theta function. To handle the problem we take the somewhat nonstandard tack of working in the framework of Weil's old book [Weil VK] on Kähler varieties. This is advantageous because Weil's austere conceptual approach to theta functions obviates quite a lot of bookkeeping. For example, there is no need to choose A- and B-cycles, and hence there is a generally lower risk of making a sign error, cf. [Mumford Tata II, p. 3.81]. In §2 below we explain our Weil-style point of view on classical theta identities. In §3 below we fit the quartic and determinantal theta identities together within the Weil framework, thereby obtaining our main results.

2. A theta function toolkit

We review what we need of the general theory of theta functions, following [Weil VK]. Then we work through a series of examples in order to bring notions treated in [Fay], [Griffiths-Harris], [Mumford Tata I], [Mumford Tata II] and [Mumford Tata III] into the Weil picture.

2.1. **A general theta formalism.**

2.1.1. *The setting.* Fix a compact complex manifold V of Kähler type. (In applications V is going to be an abelian variety, or a compact Riemann surface, or a product of such.) Fix a universal covering map $\tilde{V} \to V$ (we do not bother to give the map a name) and denote its automorphism group by G. The group law in G is by definition composition of functions and hence G acts naturally on the left of $\tilde{V}$. Given $\tilde{v} \in \tilde{V}$ and its image $v \in V$ under the covering map $\tilde{V} \to V$, we call $\tilde{v}$ a *lifting* of v. Analogously we speak of liftings of paths. Given a differential form ω on V and its pull-back $\tilde{\omega}$ via the covering map $\tilde{V} \to V$, we call $\tilde{\omega}$ the *lifting* of ω. Given a closed 1-form ζ on V and a function z on $\tilde{V}$ such that dz is the lifting of ζ, we call z a *primitive* of ζ.

2.1.2. *Systems of multipliers.* We call a family $\{F_\sigma\}$ of nowhere-vanishing holomorphic functions on $\tilde{V}$ indexed by $\sigma \in G$ a *system of multipliers* under the following conditions:

- $F_{\sigma\tau} = \tau^* F_\sigma \cdot F_\tau$ for all $\sigma, \tau \in G$.
- $d \log F_\sigma$ is the lifting of a holomorphic 1-form on V for all $\sigma \in G$.

When the roles of $\tilde{V}$ and G require emphasis, we say that $\{F_\sigma\}$ is a system on multipliers *on $\tilde{V}$ relative to G*. If all the functions F_σ are constants, we say that $\{F_\sigma\}$ is a system of *constant* multipliers. A system of constant multipliers is simply a homomorphism from G to the group of nonzero complex numbers. If all the functions $|F_\sigma|$ are identically equal to 1, we say that $\{F_\sigma\}$ is *unitary*. By the Maximum Principle a unitary system of multipliers is a system of constant multipliers. We say that systems of multipliers $\{F_\sigma\}$ and $\{F'_\sigma\}$ are *equivalent* if for some nowhere-vanishing holomorphic function u on $\tilde{V}$ we have $F'_\sigma / F_\sigma = \sigma^* u / u$ for all $\sigma \in G$. By Hodge theory every system of constant multipliers is equivalent to a unitary system of multipliers. Moreover, distinct unitary systems of multipliers are inequivalent.

2.1.3. *Theta functions.* A not-identically-vanishing meromorphic function ϑ on $\tilde{V}$ transforming for some system of multipliers $\{F_\sigma\}$ by the rule $\sigma^*\vartheta = F_\sigma \cdot \vartheta$ for all $\sigma \in G$ is called a *theta function.* When the roles of $\tilde{V}$ and G require emphasis, we say that ϑ is a theta function *on* $\tilde{V}$ *relative to* G. Given a theta function ϑ transforming according to a system of multiplier $\{F_\sigma\}$, we say that ϑ *determines* the system of multipliers $\{F_\sigma\}$; clearly $\{F_\sigma\}$ is uniquely determined by ϑ. The divisor of a theta function ϑ is G-invariant and hence descends to a divisor of V, say D; in this situation we say that ϑ *represents* D, that ϑ is *effective* if D is effective, and that ϑ is *trivial* if $D = 0$. We say that theta functions are *equivalent* if they represent the same divisor of V. Equivalent theta functions determine equivalent multiplier systems. By the Maximum Principle an effective theta function with G-invariant absolute value is constant.

2.1.4. *The first Chern class of a divisor.* Fix a divisor D of V. For the purpose of checking signs and factors of 2 and π, we briefly recall the method used in [Weil VK, Chap. 5] for obtaining the first Chern class $c_1(D)$ in the de Rham cohomology of V. Fix an open covering $\{U_i\}$ of V by nonempty open sets such that that D restricted to U_i is the divisor of a meromorphic function f_i on U_i. For indices i and j such that $U_i \cap U_j \neq \emptyset$, let F_{ij} be the unique nowhere-vanishing holomorphic function on $U_i \cap U_j$ such that

$$f_j|_{U_i\cap U_j} = F_{ij} \cdot f_i|_{U_i\cap U_j}.$$

The family $\{F_{ij}\}$ thus defined is called the *system of transition functions* associated to $\{U_i\}$ and $\{f_i\}$. By a method recalled in the course of the proof of Proposition 2.1.6 below, it is possible to construct for each i a smooth $(1,0)$-form η_i on U_i such that

$$\frac{1}{2\pi i} d\log F_{ij} = \eta_j|_{U_i\cap U_j} - \eta_i|_{U_i\cap U_j}$$

whenever $U_i \cap U_j \neq \emptyset$; any such family $\{\eta_i\}$ will be called a *connection* for $\{F_{ij}\}$. Given any connection $\{\eta_i\}$ for $\{F_{ij}\}$, there exists a unique smooth closed 2-form α on V described locally by the conditions

$$\alpha|_{U_i} = d\eta_i.$$

The closed 2-form α is called the *curvature* of the connection $\{\eta_i\}$. The de Rham cohomology class of the 2-form α depends only on D, not on the intervening choices, and this class is none other than $c_1(D)$.

Lemma 2.1.5. *Let ϕ be a smooth compactly supported function on $\mathbb{C}^n$. Let β be a smooth closed $(2n-2)$-form defined on an open set $U \subset \mathbb{C}^n$ containing the support of ϕ. The integral $\int_{U\setminus\{z_1=0\}} d\phi \wedge d\log z_1 \wedge \beta$ is absolutely convergent and equals $-2\pi i \int_{U\cap\{z_1=0\}} \phi\beta$.*

Proof. Sophomore calculus. □

Proposition 2.1.6. *Let n be the complex dimension of V. Let D be a complex submanifold of V of codimension 1. A closed 2-form α on V belongs to the cohomology class $c_1(D)$ if and only if $\int_V \alpha \wedge \beta = \int_D \beta$ for all closed $(2n-2)$-forms β on V.*

Proof. By Poincaré duality it suffices merely to exhibit a closed 2-form α belonging to the class $c_1(D)$ such that $\int_V \alpha \wedge \beta = \int_D \beta$ for all closed $(2n-2)$-forms β on V, and it is well known in principle how to do this. We take care over the details just for the purpose of checking signs and factors of 2 and π. By hypothesis we can choose a finite open covering $\{U_i\}$ of V by coordinate patches and for each i a holomorphic function f_i on U_i belonging to some coordinate system on U_i such that $D \cap U_i = \{f_i = 0\}$. Let $\{F_{ij}\}$ be the system of transition functions associated to $\{U_i\}$ and $\{f_i\}$. Fix a partition of unity $\{\phi_i\}$ subordinate to $\{U_i\}$. For all indices i and ℓ there exists a unique smooth $(1,0)$-form $\eta_{i\ell}$ on U_i such that

$$2\pi i \cdot \eta_{i\ell}|_{U_i \cap U_\ell} = \phi_\ell|_{U_i \cap U_\ell} \cdot d\log F_{\ell i} \text{ if } U_i \cap U_\ell \neq \emptyset, \quad \eta_{i\ell}|_{U_i \setminus \operatorname{supp} \phi_\ell} = 0.$$

For all indices ℓ there exists a unique smooth $(1,0)$-form ζ_ℓ on $V \setminus \operatorname{supp} D$ such that

$$\zeta_\ell|_{U_\ell \setminus \operatorname{supp} D} = \phi_\ell|_{U_\ell \setminus \operatorname{supp} D} \cdot d\log f_\ell|_{U_\ell \setminus \operatorname{supp} D}, \quad \zeta_\ell|_{V \setminus (\operatorname{supp} D \cup \operatorname{supp} \phi_\ell)} = 0.$$

The family $\{\sum_\ell \eta_{i\ell}\}$ is then connection for $\{F_{ij}\}$ with curvature α satisfying the relation

$$-2\pi i \cdot \alpha|_{V \setminus D} = \sum_\ell d\zeta_\ell.$$

The closed 2-form α has the desired property by Lemma 2.1.5. □

2.1.7. *Weil gauges.* We say that a real-valued function Φ on $\tilde{V}$ is a *Weil gauge* if of the form $\Phi = \sum_{i=1}^N c_i |z_i|^2$ where the z_i are primitives of holomorphic 1-forms on V and the c_i are real constants. The notion of Weil gauge plays a key if anonymous role in [Weil VK, Chap. V].

Theorem 2.1.8. *Fix a divisor D of V and a Weil gauge Φ on $\tilde{V}$. The following conditions are equivalent:*

1. $\frac{i}{2}\partial\bar{\partial}\Phi$ *is the lifting of a closed real $(1,1)$-form on V belonging to the first Chern class $c_1(D)$.*
2. *There exists a theta function ϑ unique up to a nonzero constant factor such that ϑ represents D and $e^{-\pi\Phi}|\vartheta|^2$ is G-invariant.*

(If the first condition holds we say that Φ is a gauge *for D; Proposition 2.1.6 is sometimes convenient for checking this. Under the condition that $e^{-\pi\Phi}|\vartheta|^2$ is G-invariant we say that ϑ is* Φ-normalized. *The*

ratio of any two Φ-normalized theta functions necessarily transforms according to a unitary character of G.)

2.1.9. *Partial sketch of proof.* The Maximum Principle proves uniqueness. The proofs of the implications (1⇒2) and (2⇒1) are more or less evident modifications of the proofs of [Weil VK, Thm. 2, Chap. V] and [Weil VK, Prop. 3, Chap. V], respectively. We therefore need not provide a detailed proof of the equivalence (1⇔2). Still, since we have superficially modified Weil's theory by stressing the notion of Weil gauge, we are under some obligation at least to check signs and factors of 2 and π. We therefore compromise: we very rapidly sketch a proof of the implication (2⇒1) and then omit further details.

Let $\{F_\sigma\}$ be the system of multipliers determined by ϑ. By hypothesis we have

$$|F_\sigma|^2 = e^{\pi(\sigma^*\Phi - \Phi)}$$

and hence

$$d\log F_\sigma = \pi\partial(\sigma^*\Phi - \Phi)$$

for all $\sigma \in G$. Let $\{U_i\}$ be a finite covering of V by geodesically convex nonempty open sets. For each i fix a section $s_i : U_i \to \tilde{V}$ of the covering map. Let $\{F_{ij}\}$ be the system of transition functions associated to $\{U_i\}$ and $\{s_i^*\vartheta\}$—meaning if $U_i \cap U_j \neq \emptyset$, then we have

$$s_j^*\vartheta\mid_{U_i\cap U_j} = F_{ij}\cdot s_i^*\vartheta\mid_{U_i\cap U_j}.$$

Whenever $U_i \cap U_j \neq \emptyset$, choose $\sigma_{ij} \in G$ to satisfy the condition

$$s_j|_{U_i\cap U_j} = \sigma_{ij}\circ s_i|_{U_i\cap U_j}.$$

Such a choice exists and is unique because $U_i \cap U_j$ is geodesically convex and *a fortiori* connected. Then whenever $U_i \cap U_j \neq \emptyset$ we have

$$\begin{aligned}
\tfrac{1}{2\pi i}d\log F_{ij} &= \tfrac{1}{2\pi i}s_i^* d\log F_{\sigma_{ij}}|_{U_i\cap U_j} \\
&= \tfrac{1}{2i}s_i^*\partial(\sigma_{ij}^*\Phi - \Phi)|_{U_i\cap U_j} \\
&= \tfrac{1}{2i}s_j^*\partial\Phi\mid_{U_i\cap U_j} - \tfrac{1}{2i}s_i^*\partial\Phi\mid_{U_i\cap U_j},
\end{aligned}$$

i. e., $\left\{\frac{1}{2i}s_i^*\partial\Phi\right\}$ is a connection for $\{F_{ij}\}$. Therefore the real closed $(1,1)$-form α defined locally by the conditions

$$\alpha\mid_{U_i} = \frac{i}{2}s_i^*\partial\bar{\partial}\Phi$$

belongs to the class $c_1(D)$. □

2.1.10. *Complement.* With an emphasis on line bundles rather than on divisors, Theorem 2.1.8 takes the following form. Let $\mathcal{E}$ be a line bundle on V with pullback $\tilde{\mathcal{E}}$ to $\tilde{V}$. Let Φ be a Weil gauge on $\tilde{V}$. Then the following conditions are equivalent:

(1) $-\frac{i}{2}\partial\bar{\partial}\Phi$ is the lifting of a closed real $(1,1)$-form on V belonging to the first Chern class of the line bundle $\mathcal{E}$.
(2) There exists a multiplier system $\{F_\sigma\}$ such that for some global trivialization $\tilde{e}$ of $\tilde{\mathcal{E}}$ we have

$$\sigma^*\tilde{e} = F_\sigma \cdot \tilde{e}, \quad |F_\sigma|^2 = e^{\pi(\sigma^*\Phi - \Phi)}$$

for all $\sigma \in G$, and moreover the multiplier system $\{F_\sigma\}$ thus attached to $\mathcal{E}$ and Φ is unique.

Further, with $\mathcal{E}$, Φ and $\{F_\sigma\}$ as above, if $\mathcal{E} = \mathcal{O}_V(-D)$ for some divisor D, then Φ is a gauge for D and $\{F_\sigma\}$ is the multiplier system determined by any Φ-normalized theta function representing D.

2.2. **Example: Principally polarized complex tori.**

2.2.1. *Definition.* Following [Weil VK, Chap. VI], we define a *principally polarized complex torus* of complex dimension g to be a triple

$$(W, H, \Lambda)$$

consisting of

- a g-dimensional complex vector space W,
- a positive definite hermitian form

$$H : W \times W \to \mathbb{C}$$

  anti-linear on the left and
- a cocompact discrete subgroup $\Lambda \subset W$

such that

- the relations

$$\Im H(A_i, B_j) = \delta_{ij}, \quad \Im H(A_i, A_j) = 0 = \Im H(B_i, B_j)$$

  hold for at least one $\mathbb{Z}$-basis $\{A_i, B_i\}_{i=1}^g$ of Λ.

Any $\mathbb{Z}$-basis for Λ with the special property above we call *symplectic*. The hermitian form H naturally gives rise to an invariant Kähler metric on the complex Lie group W/Λ.

2.2.2. *Specialization of the setting.* Fix a principally polarized complex torus (W, H, Λ) of complex dimension g. We now specialize the setting of §2.1.1 as follows:

$$V = W/\Lambda, \quad \tilde{V} = W, \quad G = \{w \mapsto \lambda + w \mid \lambda \in \Lambda\}.$$

Also we put

$$\Phi(w) := H(w, w)$$

for all $w \in W$, thereby defining a Weil gauge. In the next several paragraphs we recall how to classify and to construct explicitly all Φ-normalized effective theta functions. We also recall the Riemann quartic theta identity in a convenient form.

2.2.3. *Semicharacters.* A function ψ on Λ taking values in the group of complex numbers of absolute value 1 is called a *semicharacter* of Λ with respect to H if

$$\psi(\lambda + \mu) = \psi(\lambda)\psi(\mu) \exp\left(\pi i \Im H(\lambda, \mu)\right)$$

for all $\lambda, \mu \in \Lambda$. The square of a semicharacter is a unitary character, whence the terminology; also the ratio of any two semicharacters is a unitary character. Real semicharacters play an especially important role in the sequel, and these have the following explicit description. Fix a symplectic $\mathbb{Z}$-basis $\{A_i, B_i\}_{i=1}^{g}$ for Λ arbitrarily. Let A (resp., B) be the vector of length g with entries A_i (resp., B_i). Every real semicharacter ψ of Λ with respect to H takes the form

$$\psi(m \cdot A + n \cdot B) = (-1)^{m\cdot n + a\cdot m + b\cdot n} \quad (m, n \in \mathbb{Z}^g)$$

for some $a, b \in \mathbb{Z}^g$ uniquely determined modulo 2. It can be shown (for an indication of proof see the end of §2.2.6 below) that the parity of the inner product $a \cdot b$ depends only on the real semicharacter ψ, not on the choice of symplectic $\mathbb{Z}$-basis $\{A_i, B_i\}$. We define the *parity* of ψ to be that of $a \cdot b$.

2.2.4. *Classification of Φ-normalized effective theta functions.* According to [Weil VK, Chap. 6], for each semicharacter ψ of Λ with respect to H there exists a not-identically-vanishing entire function ϑ on W unique up to a nonzero constant factor such that

$$\vartheta(w + \lambda) = \psi(\lambda) \exp\left(\pi H(\lambda, w) + \frac{\pi}{2} H(\lambda, \lambda)\right) \vartheta(w)$$

for all $w \in W$ and $\lambda \in \Lambda$. (We briefly sketch in §2.2.6 below the calculation by which existence and uniqueness are proved.) We call any such function ϑ a *theta function* of *type* (W, H, Λ, ψ). Clearly a theta function of type (W, H, Λ, ψ) is a Φ-normalized effective theta function in the sense of Theorem 2.1.8. Conversely, every Φ-normalized effective theta function ϑ in the sense of Theorem 2.1.8 is a theta function of type (W,H,Λ,ψ) for a uniquely determined semicharacter ψ.

2.2.5. *Natural operations on theta functions.* Let a theta function $\vartheta(w)$ of type (W,H,Λ,ψ) be given. Then $\vartheta(-w)$ is a theta function of type $\big(W,H,\Lambda,\overline{\psi}\big)$. In particular, if $\overline{\psi}=\psi$, then $\vartheta(\pm w)=\pm\vartheta(w)$. In other words, if ψ is real, then ϑ has a well defined parity. Given another semicharacter ψ' of Λ with respect to H and $t\in W$ such that

$$\psi'(\lambda)=\psi(\lambda)\exp(2\pi i\Im H(\lambda,t))$$

for all $\lambda\in\Lambda$, then

$$\exp(-\pi H(t,w))\vartheta(w+t)$$

is a theta function of type (W,H,Λ,ψ'). In other words, roughly speaking, any given Φ-normalized theta function gives rise to all others by translation and adjustment by elementary nowhere-vanishing factors.

2.2.6. *Explicit construction of theta functions.* Fix a symplectic $\mathbb{Z}$-basis $\{A_i,B_i\}_{i=1}^g$ for Λ and a semicharacter ψ of Λ with respect to H. For simplicity we assume that:

- $W=\mathbb{C}^g$, the latter viewed for computational purposes as the space of column vectors of length g with complex entries.
- A_i is the i^{th} column of the g by g identity matrix.

Let τ be the g by g matrix defined by the following condition:

- B_i is the i^{th} column of τ.

In this situation necessarily:

- τ is symmetric with positive definite imaginary part.
- $\bar{v}^T(\Im\tau)^{-1}w=H(v,w)$ for all $v,w\in\mathbb{C}^g$.

Moreover, there exist column vectors $a,b\in\mathbb{R}^g$ unique modulo $\mathbb{Z}^g$ with the following property:

- $\psi(m+\tau n)=\exp\big(\pi i\, m^Tn+2\pi i(m^Ta-n^Tb)\big)$ for all $m,n\in\mathbb{Z}^g$.

As usual, cf. [Fay, p. 1] or [Mumford Tata I, p. 123], put

$$\theta\left[\begin{array}{c}a\\ b\end{array}\right](w,\tau):=\sum_{\ell\in\mathbb{Z}^g}\exp\big(\pi i(\ell+a)^T\tau(\ell+a)+2\pi i(\ell+a)^T(w+b)\big)$$

thereby defining a holomorphic function of $w\in\mathbb{C}^g$ that does not vanish identically. To abbreviate we now drop reference to a, b and τ since these are being held fixed, and we simply write $\theta(w)$. We have

$$\theta(w+m+\tau n)=\exp\big(2\pi i(a^Tm-b^Tn)-\pi in^T\tau n-2\pi in^Tw\big)\,\theta(w)$$

for all $m,n\in\mathbb{Z}^g$ and $w\in\mathbb{C}^g$. It is easy to show by the method of "undetermined Fourier coefficients" that the system of functional equations above characterizes $\theta(w)$ uniquely up to a nonzero constant factor. By a straightforward calculation it can be verified that the function

$$\exp\left(\frac{\pi}{2}w^T(\Im\tau)^{-1}w\right)\theta(w)$$

is a theta function of type (W, H, Λ, ψ) and moreover the only such up to a nonzero constant factor. From the explicit presentation of theta functions of type (W, H, Λ, ψ) just recalled it follows in particular that if ψ is real, then the parity of ψ as defined at the end of §2.2.3 coincides with the parity of any theta function of type (W, H, Λ, ψ).

Proposition 2.2.7. *Fix a semicharacter ψ on Λ with respect to H and a theta function ϑ of type (W, H, Λ, ψ). Let $\Lambda' \subset W$ be a cocompact discrete subgroup of W such that (W, H, Λ') is also a principally polarized abelian variety. Assume further that $\#\left(\frac{\Lambda+\Lambda'}{\Lambda\cap\Lambda'}\right) < \infty$. Fix a semicharacter ψ' of Λ' with respect to H agreeing with ψ on $\Lambda\cap\Lambda'$ and a theta function ϑ' of type (W, H, Λ', ψ'). Fix a (necessarily finite) set of representatives $L \subset \Lambda$ for the quotient $\Lambda/(\Lambda\cap\Lambda')$. Then there exists a unique family $\{C_\lambda\}_{\lambda\in L}$ of complex constants such that*

$$\vartheta(w) = \sum_{\lambda\in L} C_\lambda \exp(-\pi H(\lambda, w))\vartheta'(\lambda + w)$$

for all $w \in W$. Morever, none of the constants C_λ vanish.

Proof. We bring in a powerful idea developed at length in [Mumford Tata III]. Put

$$\mathcal{H} := \{[\lambda, s] \mid \lambda \in W, \quad s \in \mathbb{C}, \quad |s| = 1\},$$

and equip $\mathcal{H}$ with a group law by the rule

$$[\lambda, s][\mu, t] := [\lambda + \mu, st\exp(\pi i\Im H(\lambda, \mu))],$$

thereby constructing the *Heisenberg group* naturally associated to the pair (W, H). The group $\mathcal{H}$ acts naturally on the space of entire functions defined on W by the rule

$$([\lambda, s]f)(w) := s\exp\left(-\pi H(\lambda, w) - \frac{\pi}{2}H(\lambda, \lambda)\right)f(w + \lambda).$$

By definition of a semicharacter, the map

$$(\lambda \mapsto [\lambda, \psi(\lambda)]) : \Lambda \to \mathcal{H}$$

is an injective group homomorphism. We denote the image of this map by $\mathcal{H}(\Lambda, \psi)$. For any unitary character χ of Λ, a theta function of type $(W, H, \Lambda, \psi\chi)$ is the same thing as a not-identically-vanishing holomorphic function φ on W tranforming under the action of $\mathcal{H}(\Lambda, \psi)$ by the rule

$$[\lambda, \psi(\lambda)]\varphi = \chi(\lambda)\cdot\varphi.$$

Let Θ be the space of holomorphic functions on W fixed under the action of the group

$$\mathcal{H}(\Lambda,\psi)\cap\mathcal{H}(\Lambda',\psi')=\{[\lambda,\psi(\lambda)]\mid\lambda\in\Lambda\cap\Lambda'\}\subset\mathcal{H}.$$

Then, so we claim, Θ is a regular complex representation of the finite abelian group

$$\mathcal{H}(\Lambda',\psi')/(\mathcal{H}(\Lambda,\psi)\cap\mathcal{H}(\Lambda',\psi'))$$

the isotypical components of which are permuted simply transitively by the finite abelian group

$$\mathcal{H}(\Lambda,\psi)/(\mathcal{H}(\Lambda,\psi)\cap\mathcal{H}(\Lambda',\psi')).$$

Since the proof of the claim is merely a recapitulation of themes from the proof of the Stone-von Neumann theorem (see [Mumford Tata III, Thm. 1.2, p.3]), we omit the details. The claim granted, the result immediately follows. □

Corollary 2.2.8. *Fix a real semicharacter ψ_0 of Λ with respect to H. Fix a theta function ϑ_0 of type (W,H,Λ,ψ_0). Fix a set of representatives $0\in M\subset\frac{1}{2}\Lambda$ for the quotient $\frac{1}{2}\Lambda/\Lambda$. Put*

$$\vartheta_\mu(w):=\exp(-\pi H(\mu,w))\vartheta_0(w+\mu)$$

for all $\mu\in M$ and $w\in W$. Put

$$T:=\frac{1}{2}\left[\begin{array}{rrrr}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{array}\right].$$

Then there exists a unique family $\{C_\mu\}_{\mu\in M}$ of complex constants such that

$$\prod_{i=1}^4\vartheta_0\left(w_i\right)=\sum_{\mu\in M}C_\mu\prod_{i=1}^4\vartheta_\mu\left(\sum_{j=1}^4 T_{ij}w_j\right)$$

for all $w_1,w_2,w_3,w_4\in W$. Moreover, none of the constants C_μ vanish. (This result is a translation into Weil-style language of Riemann's quartic theta identity. For a classical-style presentation of the latter, see [Mumford Tata I, p. 212]*.)*

Proof. It is a tedious job but not an especially difficult one to verify that Proposition 2.2.7 applies with

$$\left[\begin{array}{c}W\\W\\W\\W\end{array}\right],\quad\left[\begin{array}{cccc}H&&&\\&H&&\\&&H&\\&&&H\end{array}\right],\quad T\left[\begin{array}{c}\Lambda\\\Lambda\\\Lambda\\\Lambda\end{array}\right],\quad\left[\begin{array}{c}\Lambda\\\Lambda\\\Lambda\\\Lambda\end{array}\right]$$

in place of W, H, Λ and Λ', respectively, the semicharacters

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \mapsto \prod_{i=1}^{4} \psi_0\left(\sum_{j=1}^{4} T_{ij}\lambda_j\right), \qquad \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \mapsto \prod_{i=1}^{4} \psi_0(\lambda_i)$$

in place of ψ and ψ', respectively, the theta functions

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \mapsto \prod_{i=1}^{4} \vartheta_0\left(\sum_{j=1}^{4} T_{ij}w_j\right), \qquad \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \mapsto \prod_{i=1}^{4} \vartheta_0\left(w_i\right)$$

in place of ϑ and ϑ', respectively, and the set

$$\left\{ \begin{bmatrix} \mu \\ \mu \\ \mu \\ \mu \end{bmatrix} \,\middle|\, \mu \in M \right\}$$

in place of L. Accordingly, there exists a unique family $\{C_\mu\}_{\mu\in M}$ of complex constants such that

$$\prod_{i=1}^{4} \vartheta_0\left(\sum_{j=1}^{4} T_{ij}w_j\right) = \sum_{\mu\in M} C_\mu \prod_{i=1}^{4} \vartheta_\mu\left(w_i\right)$$

for all $w_1, w_2, w_3, w_4 \in W$ and moreover none of the constants C_μ vanish. Since $T^2 = 1$, this last identity is equivalent to the desired one. □

2.3. **Example: Compact Riemann surfaces.**

2.3.1. *Specialization of the setting.* Let X be a compact Riemann surface of genus $g > 0$, let $\tilde{X} \to X$ be a universal covering map, and put $\Gamma := \mathrm{Aut}(\tilde{X}/X)$. We now specialize the setting of §2.1.1 to the case

$$V = X, \quad \tilde{V} = \tilde{X}, \quad G = \Gamma.$$

It is convenient to fix a basepoint $\infty \in X$ and a lifting $\tilde{\infty} \in \tilde{X}$ thereof. In the next several paragraphs we work out a Weil-style analytic description of the Jacobian of X and then we study the multiplier systems associated to theta functions on $\tilde{X}$ relative to Γ representing divisors on X of degree 0 and of degree g.

2.3.2. *Basic topological notation.* Given points $\tilde{P}, \tilde{Q} \in \tilde{X}$, we denote by $[\tilde{P} \to \tilde{Q}]$ a choice of path in X admitting a lifting to a path issuing from $\tilde{P}$ and terminating at $\tilde{Q}$. Given 1-cycles c_1 and c_2 on X in general position, let $\#(c_1 \cap c_2)$ denote the signed number of intersections of c_1 with c_2, where, as usual, we count $+1$ where c_2 crosses c_1 from right to left and -1 at the other crossings. Whenever we speak of paths, loops, cycles, chains, etc., it is understood that all such are sufficiently differentiable to integrate over.

Proposition 2.3.3. *Let W be the $\mathbb{C}$-linear dual of the space of differentials of the first kind on X. Let Λ be the subgroup of W consisting of $\mathbb{C}$-linear functionals of the form $\left(\omega \mapsto \int_c \omega\right)$ for some 1-cycle c on X. There exists a unique hermitian form*

$$H : W \times W \to \mathbb{C}$$

antilinear on the left such that for all 1-cycles c_1 and c_2 on X in general position we have

$$\Im H\left(\left(\omega \mapsto \int_{c_1} \omega\right), \left(\omega \mapsto \int_{c_2} \omega\right)\right) = \#(c_1 \cap c_2).$$

The triple (W, H, Λ) is a principally polarized abelian variety. (The triple (W, H, Λ) is the Jacobian of X described in Weil-style language. In the sequel we work exclusively with this version of the Jacobian.)

Proof. By Hodge theory the subgroup Λ is cocompact and discrete. Uniqueness of H is clear. The intersection pairing on the 1-dimensional homology of X is well known to be alternating and to have unit pfaffian. We have only to prove that H with the desired property exists and is positive definite. We construct a candidate H_0 for H as follows. To each $w \in W$ we associate a holomorphic 1-form ζ_w by the rule $w = \left(\omega \mapsto \frac{1}{2} \int \omega \wedge \bar{\zeta}_w\right)$. The map $w \mapsto \zeta_w$ identifies W in $\mathbb{C}$-antilinear fashion with the space of holomorphic 1-forms on X. Put

$$H_0(v, w) = \frac{i}{2} \int \zeta_v \wedge \bar{\zeta}_w$$

for all $v, w \in W$, thereby defining a positive definite hermitian form H_0 on W antilinear on the left. For each 1-cycle c on X there exists by Poincaré duality and Hodge theory a unique holomorphic 1-form ζ_c on X such that

$$\int_c \alpha = \int \alpha \wedge \Re\zeta_c = \frac{1}{2} \int \alpha \wedge \bar{\zeta}_c$$

for all smooth closed 1-forms α on X. It follows that for all 1-cycles c_1 and c_2 on X in general position we have

$$\begin{aligned}&\Im H_0\left(\left(\omega\mapsto\int_{c_1}\omega\right),\left(\omega\mapsto\int_{c_2}\omega\right)\right)\\ =\;&\Im\left(\frac{i}{2}\int\zeta_{c_1}\wedge\bar{\zeta}_{c_2}\right)=\int\Re\zeta_{c_1}\wedge\Re\zeta_{c_2}=\#(c_1\cap c_2).\end{aligned}$$

Thus our candidate H_0 has all the desired properties, i. e., $H=H_0$. □

2.3.4. *Convenient abuses of notation.* Hereafter we often treat 1-chains as though they were points of W. More precisely, given a 1-chain c, we often just write c where more properly we should write, say, $\left(\omega\mapsto\int_c\omega\right)$. Further in this line we also often treat elements of Γ as if they were points of W. More precisely, given $\sigma\in\Gamma$, we often just write σ where more properly we should write, say, $\left(\omega\mapsto\int_{[\tilde{\infty}\to\sigma\tilde{\infty}]}\omega\right)$. This saves a lot of writing and should not cause any confusion.

Proposition 2.3.5. *Let a differential ω of the first kind on X and $w\in W$ be given. If $\int_c\omega=H(c,w)$ for all 1-cycles on X, then $\omega=0$ and $w=0$.*

Proof. We continue in the setting of the proof of Proposition 2.3.3. We have

$$\int_c\bar{\omega}=H(w,c)=\frac{i}{2}\int\zeta_w\wedge\bar{\zeta}_c=i\int\zeta_w\wedge\Re\zeta_c=\int_c i\zeta_w,$$

hence the 1-forms $\bar{\omega}$ and $i\zeta_w$ have the same periods and hence they are equal. But the latter 1-form is holomorphic and the former antiholomorphic. Therefore both must vanish. □

Proposition 2.3.6. *Let D be a divisor of X of degree zero.*

(1) *There exists a theta function ϑ on $\tilde{X}$ relative to Γ unique up to a nonzero constant factor such that ϑ represents D and tranforms under the action of Γ by a unitary character.*

(2) *With ϑ as above, we have*

$$\sigma^*\vartheta=\exp\left(2\pi i\Im H(\partial^{-1}D,\sigma)\right)\cdot\vartheta$$

for all $\sigma\in\Gamma$ and 1-chains $\partial^{-1}D$ on X with boundary D.

(3) *Every unitary character of Γ thus appears in association with some divisor of X of degree zero.*

(The proposition is a restatement of the theorems of Abel and Jacobi in Weil-style language.)

Proof. Statement 1 is equivalent to the classical fact that there exists a unique differential ξ of the third kind on X with residual divisor D and pure imaginary periods. (Of course statement 1 is also a very special case of Theorem 2.1.8.) Statements 1 and 2 granted, statement 3 is proved by a well known argument we need not repeat. It remains only to prove statement 2, and this is just a matter of translating from classical-style language to Weil-style language. We take care with the details in order to check signs and factors of 2 and π.

Clearly $d\log\vartheta$ is the lifting of ξ and we have

$$\sigma^*\vartheta = \exp\left(\int_{[\tilde{\infty}\to\sigma\tilde{\infty}]}\xi\right)\vartheta$$

for all $\sigma\in\Gamma$, where the loop $[\tilde{\infty}\to\sigma\tilde{\infty}]$ is chosen to avoid the support of D. Now arbitrarily fix a 1-chain c_D with boundary D. In order to prove statement (ii), it suffices to verify that

$$\int_c \xi \equiv 2\pi i\Im H(c_D, c) \bmod 2\pi i\mathbb{Z}$$

for all loops c on X avoiding the support of D.

In the usual way cut X open to form a $4g$-sided polygon and construct a homology basis $\{A_i, B_i\}_{i=1}^g$ in the standard configuration—meaning in particular that

$$\#(A_i\cap A_j) = 0 = \#(B_i\cap B_j) = 0,\quad \#(A_i\cap B_j) = \delta_{ij}$$

for $i,j = 1,\dots,g$. We may assume without loss of generality that c_D is contained in the interior of the polygon, and that c is one of the A's and B's. According to the classical reciprocity law for differentials of first and third kinds (see [Griffiths-Harris, p. 230]) we have

$$\sum_{i=1}^g\left(\left(\int_{A_i}\omega\right)\left(\int_{B_i}\xi\right)-\left(\int_{B_i}\omega\right)\left(\int_{A_i}\xi\right)\right) = 2\pi i\int_{c_D}\omega$$

for all differentials ω of the first kind on X. In other words, we have an identity

$$\sum_{i=1}^g\left(\left(\frac{1}{2\pi i}\int_{B_i}\xi\right)A_i-\left(\frac{1}{2\pi i}\int_{A_i}\xi\right)B_i\right) = c_D$$

holding in W. This last equation is enough to finish the proof. □

2.3.7. *The Abel map and associated Weil gauge.* Put

$$z(\tilde{P}) := \left(\omega \mapsto \int_{[\tilde{\infty}\to\tilde{P}]} \omega\right) \in W$$

for all $\tilde{P} \in \tilde{X}$, thereby defining the *Abel map*

$$z : \tilde{X} \to W$$

based at $\tilde{\infty}$. In calculations below we repeatedly exploit the relation

$$\sigma^* z = z + \sigma$$

for all $\sigma \in \Gamma$; of course, on the right, by abuse of notation, σ stands in for the linear functional

$$\left(\omega \mapsto \int_{[\tilde{\infty}\to\sigma\tilde{\infty}]} \omega\right) \in W.$$

Now let

$$\Phi(w) := H(w, w)$$

be the Weil gauge with which the Jacobian (W, H, Λ) is canonically equipped. Clearly the pull-back $z^*\Phi$ is a Weil gauge on $\tilde{X}$. Another more direct description of $z^*\Phi$ can be given as follows. Let $\zeta_1, \ldots, \zeta_g$ be any $\mathbb{C}$-basis for the space of differentials of the first kind on X orthonormalized by the condition

$$\frac{i}{2}\int \zeta_i \wedge \bar{\zeta}_j = \delta_{ij}.$$

Then we have

$$(z^*\Phi)(\tilde{P}) = \sum_{i=1}^{g} \left| \int_{[\tilde{\infty}\to\tilde{P}]} \zeta_i \right|^2$$

for all $\tilde{P} \in \tilde{X}$. From the latter description of Φ it follows directly that $\frac{i}{2}\partial\bar{\partial} z^*\Phi$ is the lifting of the closed real positive $(1,1)$-form

$$\frac{i}{2}\sum_{i=1}^{g} \zeta_i \wedge \bar{\zeta}_i.$$

In turn it follows by Proposition 2.1.6 that the function $z^*\Phi$ is a gauge for any divisor of degree g.

2.3.8. *Association of a semicharacter to each divisor of degree $g-1$.* Fix a divisor D of X of degree $g-1$. By Theorem 2.1.8 the divisor $D+\infty$ is represented by a $z^*\Phi$-normalized theta function on $\tilde{X}$ relative to Γ, say ϑ, unique up to a nonzero constant factor. It is easy to see that the multiplier system determined by ϑ has to be the pull-back via the Abel map z of the multiplier system determined by some Φ-normalized theta function on W relative to Λ. The upshot is that there exists a unique semicharacter ψ_D of Λ with respect to H such that

$$\sigma^*\vartheta = \psi_D(\sigma)\cdot\exp\left(\pi H(\sigma,z)+\frac{\pi}{2}H(\sigma,\sigma)\right)\cdot\vartheta$$

for all $\sigma\in\Gamma$.

Proposition 2.3.9. *Notation as in the paragraph above, the following hold:*

(1) *The semicharacter ψ_D is independent of the choice of the base-point ∞ and lifting $\tilde{\infty}$ thereof.*

(2) *The construction $D\mapsto\psi_D$ puts the classes of divisors of X of degree $g-1$ in bijective correspondence with the semicharacters of Λ with respect to H.*

(The proposition is a Weil-style description of the correspondence between theta functions with characteristics and divisors of degree $g-1$. See [Mumford Tata I, Chap. II, §3] *for a classical-style treatment of this correspondence.)*

Proof. Statement 1 granted, statement 2 follows immediately from Proposition 2.3.6. We turn now to the proof of statement 1. Fix $P\in X$ and a lifting $\tilde{P}\in\tilde{X}$ arbitrarily. By a repetition of the arguments made above there exists a theta function ϑ_1 unique up to a nonzero constant multiple and a unique semicharacter ψ_1 of Λ with respect to H such that ϑ_1 represents the divisor $D+P$ and transforms according to the rule

$$\sigma^*\vartheta_1 = \psi_1(\sigma)\cdot\exp\left(\pi H(\sigma,z-z(\tilde{P}))+\frac{\pi}{2}H(\sigma,\sigma)\right)\cdot\vartheta_1$$

for all $\sigma\in\Gamma$. Then the theta function

$$\vartheta_2=\vartheta_1/\vartheta\cdot\exp\left(\pi H\left([\tilde{\infty}\to\tilde{P}],z-z(\tilde{P})\right)\right)$$

represents the divisor $P-\infty$ of degree zero and transforms according to the unitary rule

$$\sigma^*\vartheta_2 = (\psi_1/\psi_D)(\sigma)\cdot\exp\left(2\pi i\Im H\left([\tilde{\infty}\to\tilde{P}],\sigma\right)\right)\cdot\vartheta_2$$

for all $\sigma\in\Gamma$. By Proposition 2.3.6 it follows that $\psi_1=\psi_D$. □

2.4. **Example: The prime form.**

2.4.1. *Specialization of the setting.* Fix a compact Riemann surface X of genus $g>0$ and a universal covering map $\tilde{X}\to X$. Fix a basepoint $\infty\in X$ and lifting $\tilde{\infty}\in\tilde{X}$ thereof. Put $\Gamma:=\mathrm{Aut}(\tilde{X}/X)$. We specialize the setting of §2.1.1 to the case

$$V=X\times X,\quad \tilde{V}=\tilde{X}\times\tilde{X},\quad G=\Gamma\times\Gamma.$$

Let

$$\Delta\subset X\times X$$

be the diagonally embedded copy of X and consider the divisor

$$\Delta'=-X\times\infty-\infty\times X+\Delta.$$

We are going to find a natural choice of gauge for Δ' and then we are going to calculate the multiplier system of the correspondingly normalized theta function representing Δ'.

2.4.2. *A gauge for* Δ'. Let (W,H,Λ) be the Jacobian of X. Let

$$\Phi(w):=H(w,w)$$

be the Weil gauge naturally associated to the Jacobian. Let z denote the Abel map $\tilde{X}\to W$ based at $\tilde{\infty}$. For $i=1,2$, let $z^{(i)}$ denote the i^{th} projection $\tilde{X}\times\tilde{X}\to\tilde{X}$ followed by the Abel map z. Put

$$\Psi:=\Phi\left(z^{(1)}-z^{(2)}\right)-\Phi\left(z^{(1)}\right)-\Phi\left(-z^{(2)}\right).$$

thereby defining a Weil gauge Ψ on $\tilde{X}\times\tilde{X}$. We claim that Ψ is a gauge for Δ'. Let

$$p_1,p_2:X\times X\to X$$

be the two projections and let $\zeta_1,\ldots,\zeta_g$ be a basis for the holomorphic 1-forms on X orthonormalized by the condition

$$\frac{i}{2}\int\zeta_i\wedge\bar{\zeta}_j=\delta_{ij}.$$

Then $\frac{i}{2}\partial\bar{\partial}\Psi$ is the lifting of the real closed $(1,1)$-form

$$\begin{aligned}\alpha &= \frac{i}{2}\sum_{i=1}^{g}\left((p_1^*\zeta_i - p_2^*\zeta_i)\wedge(p_1^*\bar{\zeta}_i - p_2^*\bar{\zeta}_i) - p_1^*(\zeta_i\wedge\bar{\zeta}_i) - p_2^*((-\zeta_i)\wedge(-\bar{\zeta}_i))\right)\\ &= \frac{i}{2}\sum_{i=1}^{g}\left(-p_1^*\zeta_i\wedge p_2^*\bar{\zeta}_i + p_1^*\bar{\zeta}_i\wedge p_2^*\zeta_i\right)\end{aligned}$$

on X. By Proposition 2.1.6 the proof of the claim boils down to verifying the identity

$$\int_{X\times X}\alpha\wedge\beta = \int_{\Delta}\beta - \int_{\infty\times X}\beta - \int_{X\times\infty}\beta$$

for β ranging over the $\mathbb{C}$-basis

$$\frac{i}{2}p_1^*\zeta_i\wedge p_2^*\zeta_j,\quad \frac{i}{2}p_1^*\bar{\zeta}_i\wedge p_2^*\bar{\zeta}_j,\quad \frac{i}{2}p_1^*\zeta_i\wedge p_2^*\bar{\zeta}_j,\quad \frac{i}{2}p_1^*\bar{\zeta}_i\wedge p_2^*\zeta_j$$

and

$$\frac{i}{2}p_1^*(\zeta_1\wedge\bar{\zeta}_1),\quad \frac{i}{2}p_2^*(\zeta_1\wedge\bar{\zeta}_1)$$

for the de Rham cohomology of $X\times X$ in dimension 2. The latter calculation is straightforward and can safely be omitted. Thus the claim is proved.

2.4.3. *The prime form E.* We define the *prime form E* to be the Ψ-normalized theta function on $\tilde{X}\times\tilde{X}$ unique up to a nonzero constant factor representing the divisor Δ'. The existence of E is guaranteed by Theorem 2.1.8, and the uniqueness of E is clear. The notion of prime form defined here is nearly but not exactly the same as the notion considered in [Fay, Chap. 2] or [Mumford Tata II, pp. 3.207-3.213]. We omit discussion of the comparison, just remarking that [Mumford Tata II, Lemma 2, Chap. IIIb, §1, p. 3.211] can be used to make the prime form as defined here explicit. In any case an explicit formula will not be needed. For our purposes it suffices simply to know that E exists, is unique up to a nonzero constant factor, and has the transformation properties summarized in Proposition 2.4.5 below.

2.4.4. *Guessing the multiplier system for E.* The form in which we presented the definition of the gauge Ψ was intended to suggest the following procedure for guessing the multiplier system of E. Fix a semicharacter ψ of Λ with respect to H arbitrarily. Consider the multiplier system

$$\left\{\psi(\lambda)\exp\left(\pi H(\lambda,w)+\frac{\pi}{2}H(\lambda,\lambda)\right)\right\}$$

on W relative to Λ determined by a theta function of type (W, H, Λ, ψ). Pulling back under the map $z^{(1)}$ we obtain a multiplier system

$$\left\{\psi(\sigma_1)\exp\left(\pi H\left(\sigma_1, z^{(1)}\right) + \frac{\pi}{2}H(\sigma_1,\sigma_1)\right)\right\}$$

on $\tilde{X}\times\tilde{X}$ relative to $\Gamma\times\Gamma$. Similarly, pulling back under the map $-z^{(2)}$ we obtain a multiplier system

$$\left\{\psi(-\sigma_2)\exp\left(\pi H\left(-\sigma_2, -z^{(2)}\right) + \frac{\pi}{2}H(-\sigma_2,-\sigma_2)\right)\right\},$$

and pulling back under the map $z^{(1)}-z^{(2)}$ we obtain a multiplier system

$$\left\{\psi(\sigma_1-\sigma_2)\exp\left(\pi H\left(\sigma_1-\sigma_2, z^{(1)}-z^{(2)}\right) + \frac{\pi}{2}H\left(\sigma_1-\sigma_2,\sigma_1-\sigma_2\right)\right)\right\}.$$

After dividing the last multiplier system by the product of the first two and simplifying, we obtain a multiplier system

$$\left\{\exp\left(-\pi\left(H\left(\sigma_1, z^{(2)}\right) + H\left(\sigma_2, z^{(1)}\right) + H(\sigma_1,\sigma_2)\right)\right)\right\}$$

on $\tilde{X}\times\tilde{X}$ relative to $\Gamma\times\Gamma$ such that any theta function determining that multiplier system is necessarily Ψ-normalized. This is our guess for the multiplier system determined by E. Next, we prove the guess.

Proposition 2.4.5. *We have*

$$(\sigma_1,\sigma_2)^*E = \exp\left(-\pi\left(H(\sigma_1, z^{(2)}) + H(\sigma_2, z^{(1)}) + H(\sigma_1,\sigma_2)\right)\right)\cdot E$$

for all

$$(\sigma_1,\sigma_2)\in\Gamma\times\Gamma = \mathrm{Aut}(\tilde{X}^2/X^2),$$

cf. [Mumford Tata II, p. 3.210]. *Moreover, E is antisymmetric under exchange of factors in the product $\tilde{X}\times\tilde{X}$.*

Proof. The proof is in essence an adaptation to the present situation of the proof of the theorem of the square. We introduce the following temporary notation. We write (σ,τ) instead of (σ_1,σ_2), we denote the multiplier system claimed for E by $\{F_{\sigma\tau}\}$, and we denote the actual multiplier system of E by $\{F'_{\sigma\tau}\}$. The ratio $\{F_{\sigma\tau}/F'_{\sigma\tau}\}$ is a unitary and hence constant system of multipliers. The restricted multiplier systems

$$\left\{F_{\sigma 1}\mid_{\tilde{X}\times\tilde{\infty}}\right\}_{\sigma\in\mathrm{Aut}(\tilde{X}/X)},\quad \left\{F_{1\tau}\mid_{\tilde{\infty}\times\tilde{X}}\right\}_{\tau\in\mathrm{Aut}(\tilde{X}/X)}$$

are identically equal to 1, and the analogous remark holds for $\{F'_{\sigma\tau}\}$ since the invertible sheaf $\mathcal{O}_{X\times X}(\Delta')$ has trivial restrictions to $X\times\infty$ and $\infty\times X$. Therefore the multiplier system $\{F_{\sigma\tau}/F'_{\sigma\tau}\}$ is identically equal to 1, i. e., the prime form E transforms in the claimed fashion under the action of $\Gamma\times\Gamma$. Since exchange of factors in the product $\tilde{X}\times\tilde{X}$ preserves both Ψ and Δ', and hence can alter E only by a nonzero constant factor, E is either symmetric or antisymmetric. The sign is nailed down by considering what happens near the diagonal. $\square$

Corollary 2.4.6. *For all canonical divisors K and divisors D of degree $g-1$ on X we have*

$$\overline{\psi}_D = \psi_{K-D}.$$

(In particular, the real semicharacters of Λ with respect to H are in canonical bijective correspondence with the half-canonical divisor classes on X, cf. [Mumford Tata II, Chap. IIIa, §6]*.)*

Proof. Let

$$\delta : \tilde{X} \to \tilde{X} \times \tilde{X}$$

be the diagonal mapping. Note that

$$\delta^*\Psi = -2z^*\Phi.$$

Let $\{F_{\sigma\tau}\}$ temporarily denote the multiplier system determined by E. On the one hand, since the diagonal restriction of the invertible sheaf $\mathcal{O}_{X\times X}(-\Delta')$ is isomorphic to $\Omega_X(2\infty)$, it follows in view of the remark of §2.1.10 that the diagonally restricted multiplier system

$$\{\delta^* F_{\sigma\sigma}\}_{\sigma\in\Gamma} = \{\exp(-2\pi H(\sigma, z) - \pi H(\sigma, \sigma))\}_{\sigma\in\Gamma}$$

is that determined by a $-2z^*\Phi$-normalized theta function on $\tilde{X}$ relative to Γ representing $-K-2\infty$. On the other hand, since the reciprocal of the diagonally restricted multiplier system can be written as the product of multiplier systems associated to $z^*\Phi$-normalized theta functions on $\tilde{X}$ relative to Γ representing $D+\infty$ and $K-D+\infty$, respectively, it is clear that $\psi_D\psi_{K-D} = 1$. $\square$

2.5. **Example: Determinant identities satisfied by the Riemann theta function.**

2.5.1. *Setting and notation.* We now combine elements of all the preceding examples. Fix a compact Riemann surface X of genus $g > 0$ and a universal covering map $\tilde{X} \to X$. Put $\Gamma := \mathrm{Aut}(\tilde{X}/X)$. Fix a basepoint $\infty \in X$ and lifting $\tilde{\infty} \in \tilde{X}$ thereof. Let (W, H, Λ) be the Jacobian of X. Let z be the Abel map $\tilde{X} \to W$ based at $\tilde{\infty}$. Fix an integer $n \geq g$. Given any function f defined on $\tilde{X}$ and index $i = 1, \ldots, n$, let $f^{(i)}$ denote the result of following the i^{th} projection $\tilde{X}^n \to \tilde{X}$ by f. Let E be the prime form. Given indices $1 \leq i < j \leq n$, let $E^{(i,j)}$ denote the result of following the $(i,j)^{th}$ projection $\tilde{X}^n \to \tilde{X} \times \tilde{X}$ by E. The symbol $\propto$ placed between two expressions indicates that both expressions define not-identically-vanishing meromorphic functions on the same complex manifold agreeing up to a nonzero constant factor.

2.5.2. *A special class of theta functions.* Given a semicharacter ψ on Λ with respect to H, an integer ℓ, and a meromorphic function u on $\tilde{X}$, we say that u is a *theta function* of *type* (X, ∞, ψ, ℓ) if

$$\sigma^* u = \psi(\sigma) \cdot \exp\left(\pi H(\sigma, z) + \frac{\pi}{2} H(\sigma, \sigma)\right) \cdot u$$

for all $\sigma \in G$ and morever u has no singularities save poles of order at most $\ell - 1$ at each lifting of the basepoint ∞. The collection of all such theta functions u forms a vector space over the complex numbers. (Deviating from the convention we have followed above, in this case we do not exclude the case $u = 0$.) Note that for any divisor D of degree $g-1$, integer ℓ and $z^*\Phi$-normalized theta function ϕ representing $D+\infty$ the map

$$(f \mapsto (\text{lifting of } f) \cdot \phi)$$
$$: H^0(X, \mathcal{O}_X(D + \ell\infty)) \to \left(\begin{array}{l} \text{space of theta functions} \\ \text{of type } (X, \infty, \psi_D, \ell) \end{array}\right)$$

is bijective. Since every semicharacter ψ is of the form ψ_D for some D, and since for any such D we have by assumption

$$\deg(D + n\infty) \geq g - 1 + g > 2g - 2,$$

it follows that

$$\dim_{\mathbb{C}} \left(\begin{array}{l} \text{space of theta functions} \\ \text{of type } (X, \infty, \psi, n) \end{array}\right) = n$$

by Riemann-Roch.

Lemma 2.5.3. *Fix a semicharacter ψ on Λ with respect to H. Recall that we are assuming that $n \geq g$. There exists a not-identically-vanishing meromorphic function ϕ on $\tilde{X}^n$ unique up to a nonzero constant factor with the following properties:*

(1) *The function ϕ transforms according to the rule*

$$\sigma^* \phi = \prod_{i=1}^{n} \psi(\sigma_i) \exp\left(\pi H\left(\sigma_i, z^{(i)}\right)\right) \cdot \phi$$

for all

$$\sigma = (\sigma_i)_{i=1}^n \in \Gamma^n = \mathrm{Aut}(\tilde{X}^n / X^n).$$

In particular, ϕ is a theta function on $\tilde{X}^n$ relative to Γ^n.

(2) *The divisor*

$$-(n-1)\left(\infty \times X^{n-1} + X \times \infty \times X^{n-2} + \cdots + X^{n-1} \times \infty\right)$$

is a lower bound for the divisor of X^n represented by ϕ. In particular, ϕ is regular on $(\tilde{X} \setminus \Gamma\tilde{\infty})^n$.

(3) *The function ϕ is anti-symmetric under exchange of factors of the product $\tilde{X}^n$.*

Proof. Let $T \subset \tilde{X} \setminus \Gamma\tilde{\infty}$ be a set of cardinality n such that the map

$$(u \mapsto u\,|_T) : \left(\begin{array}{l}\text{space of theta}\\ \text{functions of}\\ \text{type } (X, \infty, \psi, n)\end{array}\right) \to \left(\begin{array}{l}\text{space of}\\ \text{functions on } T\end{array}\right)$$

map is bijective. For $i = 1, \ldots, n$, by induction on i, a meromorphic function on $\tilde{X}^n$ satisfying conditions 1 and 2 vanishes identically if it vanishes identically on $T^i \times \tilde{X}^{n-i}$. Thus we find that the map

$$(\phi \mapsto \phi\,|_{T^n}) : \left(\begin{array}{l}\text{space of functions}\\ \text{satisfying 1 and 2}\end{array}\right) \to \left(\begin{array}{l}\text{space of}\\ \text{functions on } T^n\end{array}\right)$$

is bijective. Under the latter map the space of functions satisfying conditions 1, 2 and 3 maps bijectively to the space of functions on T^n antisymmetric under exchange of factors. Clearly the latter space is one-dimensional. $\square$

Proposition 2.5.4. *Let ϑ be a theta function of type (W, H, Λ, ψ). Recall that we are assuming that $n \geq g$. Let $u_1, \ldots, u_n$ be a $\mathbb{C}$-basis for the space of theta functions of type (X, ∞, ψ, n). We have*

$$\det_{i,j=1}^{n} u_j^{(i)} \propto \vartheta\left(\sum_{i=1}^{n} z^{(i)}\right) \prod_{1 \leq i < j \leq n} E^{(i,j)}.$$

(The proposition is a Weil-style formulation of a classical determinant identity satisfied by the Riemann theta function. See [Fay, Prop. 2.16, p. 29] *for a classical-style formulation of this identity.)*

Proof. The expression on the left (resp., right) side of the claimed relation defines a not-identically-vanishing meromorphic function on $\tilde{X}^n$, and moreover this function clearly satisfies conditions 1, 2 and 3 (resp., 2 and 3) enunciated in Lemma 2.5.3. It remains only to verify that the function on the right satisfies condition 1. Fix

$$\sigma = (\sigma_i)_{i=1}^n \in \Gamma^n = \mathrm{Aut}(\tilde{X}^n / X^n)$$

arbitrarily. We have

$$\begin{aligned}
& \sigma^*\vartheta\left(\sum z^{(i)}\right) \\
= \ & \vartheta\left(\left(\sum z^{(i)}\right)+\left(\sum \sigma_i\right)\right) \\
= \ & \psi(\sum \sigma_i)\exp(\pi H(\sum \sigma_i, \sum z^{(i)}) + \tfrac{\pi}{2}H(\sum \sigma_i, \sum \sigma_i))\vartheta(\sum z^{(i)}),
\end{aligned}$$

$$\begin{aligned}
& \sigma^* E^{(i,j)} \\
= \ & ((\sigma_i,\sigma_j)^*E)^{(i,j)} \\
= \ & \exp(-\pi(H(\sigma_i, z^{(j)}) + H(\sigma_j, z^{(i)}) + H(\sigma_i,\sigma_j)))E^{(i,j)}
\end{aligned}$$

by definition of a theta function of type (W, H, Λ, ψ) and the transformation law for the prime form E enunciated in Proposition 2.4.5, respectively. This is enough to finish the proof. □

Corollary 2.5.5. *For any divisor D of X of degree $g-1$, theta function ϑ of type (W, H, Λ, ψ_D), divisor D' of X of degree 0 and 1-chain $\partial^{-1}D'$ on X with boundary D' we have*

$$\vartheta\left(\partial^{-1}D'\right) = 0 \Leftrightarrow h^0\left(D - D'\right) > 0.$$

(The proposition is a Weil-style formulation of a weak version of the Riemann Vanishing Theorem. For classical-style discussions of the latter, see [Mumford Tata I, Chap. II, §3] *or* [Griffiths-Harris, Riemann-Kempf Singularity Theorem, p. 348]*.)*

Proof. Both sides of the logical equivalence to be proved depend only on the divisor class of D'. We may therefore assume without loss of generality that for some $n \geq g$, distinct points $P_1, \ldots, P_n \in X \setminus \infty$ and corresponding liftings $\tilde{P}_1, \ldots, \tilde{P}_n \in \tilde{X}$ we have

$$D' = -n\infty + \sum_{i=1}^n P_i, \quad \partial^{-1}D' = \sum_{i=1}^n [\tilde{\infty} \to \tilde{P}_i] = \sum_{i=1}^n z(\tilde{P}_i) \in W.$$

We then have

$$h^0\left(D + n\infty - \sum_{i=1}^n P_i\right) = \dim_{\mathbb{C}}\left(\begin{array}{l}\text{space of theta functions}\\ \text{of type } (X,\infty,\psi,n)\\ \text{vanishing at } \tilde{P}_1,\ldots,\tilde{P}_n\end{array}\right)$$

by the remarks of §2.5.2 and hence

$$h^0\left(D + n\infty - \sum_{i=1}^n P_i\right) > 0 \Leftrightarrow \vartheta\left(\sum_{i=1}^n z(\tilde{P}_i)\right) = 0$$

by Proposition 2.5.4. □

2.5.6. *Remark.* We continue in the setting of Corollary 2.5.5. At full strength, the Riemann Vanishing Theorem says that $h^0(D-D')$ equals the order of vanishing of ϑ at the point $\partial^{-1}D' \in W$. For an extended discussion of the Riemann Vanishing Theorem from the classical point of view, see, e. g., [Griffiths-Harris, loc. cit.]. One can deduce the Riemann Vanishing Theorem at full strength from Proposition 2.5.4 by manipulation of symmetric functions on X^n. Such manipulations are standard in soliton theory, cf. [Segal-Wilson 1985, pp. 48-52].

Corollary 2.5.7. *Fix a half-canonical divisor D on X and a theta function ϑ of type (W, H, Λ, ψ_D). The following are equivalent:*

1. *ϑ vanishes at the origin to order at least 2.*
2. *$h^0(D) \geq 2$.*
3. *The function $\vartheta\left(z^{(1)} - z^{(2)}\right)$ on $\tilde{X} \times \tilde{X}$ vanishes identically.*

(To prove the equivalence of conditions 2 and 3 only the "weak" Riemann Vanishing Theorem is needed.)

Proof. ($1 \Leftrightarrow 2$) This equivalence is a special case of the "full strength" Riemann Vanishing Theorem.

($2 \Rightarrow 3$) If $h^0(D) > 1$, then $h^0(D-P) > 0$ for all points P of X and hence $h^0(D-P+Q) > 0$ for all points P and Q of X, whence the result via Corollary 2.5.5.

(not $2 \Rightarrow$ not 3) We may assume without loss of generality that $h^1(D) = 1$, for otherwise $\vartheta(0) \neq 0$ by Corollary 2.5.5 and there is nothing to prove. To finish the proof we now follow the proof of [Mumford Tata II, Lemma 2, p. 3.211]. Pick a point P on X arbitrarily. We have either $h^0(D+P) = 1$ or else, by Riemann-Roch,

$$h^0(D-P) = h^0(K-D-P) = h^0(D+P) - 1 = 1.$$

Existence of Q such that $h^0(D+P-Q) = 0$ in the former case and $h^0(D-P+Q) = 0$ in the latter case is clear, whence in both cases the result via Corollary 2.5.5. □

2.6. **Example: An *ad hoc* deformation theory.** We continue to work in the setting of §2.5.

Proposition 2.6.1. *Fix a divisor D of X of degree $g-1$. Fix a theta function ϑ of type (W, H, Λ, ψ_D). Make now the stronger assumption that $n \geq g+1$. Fix distinct points*

$$P_1, \ldots, P_n \in X \setminus \{\infty\}$$

and corresponding liftings

$$\tilde{P}_1, \ldots, \tilde{P}_n \in \tilde{X}$$

such that

$$\vartheta\left(\sum_{i=1}^{n} z(\tilde{P}_i)\right) \neq 0.$$

For $i = 1, \ldots, n$ *and* $\tilde{P} \in \tilde{X} \setminus \Gamma\tilde{\infty}$, *put*

$$u_i(\tilde{P}) := \vartheta\left(z(\tilde{P}) + \sum_{\alpha\in\{1,\ldots,n\}\setminus\{i\}} z(\tilde{P}_\alpha)\right) \prod_{\alpha\in\{1,\ldots,n\}\setminus\{i\}} E\left(\tilde{P}, \tilde{P}_\alpha\right),$$

thereby defining a function regular on $\tilde{X} \setminus \Gamma\tilde{\infty}$ *and meromorphic on* $\tilde{X}$. *The following hold:*

(1) *The function* u_i *is a theta function of type* (X, ∞, ψ_D, n).
(2) *The functions* $u_1, \ldots, u_n$ *form a* $\mathbb{C}$*-basis for the space of theta functions of type* (X, ∞, ψ_D, n).
(3) *The divisor of* X *represented by* u_i, *say* D_i, *belongs to the divisor class of* $D + \infty$.
(4) *The divisors* $D_1 + (n-1)\infty, \ldots, D_n + (n-1)\infty$ *are effective and have no zeroes in common.*

Proof. 1. This is clear in view of our complete knowledge of the laws governing the transformation of ϑ and E.

2. By construction of the u_i we have

$$\prod_{i=1}^{n} u_i(\tilde{P}_i) = \det_{i,j=1}^{n} u_i(\tilde{P}_j) \neq 0$$

and hence the u_i are $\mathbb{C}$-linearly independent.

3. By definition of ψ_D, a $z^*\Phi$-normalized theta function representing $D + \infty$ transforms under the action of Γ according to the same law as does any theta function of type (X, ∞, ψ_D, n).

4. It is clear that the divisors $D_i + (n-1)\infty$ are effective. But further, under our assumption that $n \geq g + 1$ we have

$$\deg(D + n\infty) \geq (g-1) + (g+1) = 2g,$$

and hence the linear system of effective divisors belonging to the divisor class of $D + n\infty$ is basepoint free. □

Proposition 2.6.2. *We continue in the setting of the proposition above. Fix* $w_0 \in W$. *Put*

$$\dot{\vartheta}(w) := \left.\frac{\partial}{\partial s} \log \vartheta\left(w + s w_0\right)\right|_{s=0},$$

thereby defining a meromorphic function on W regular away from the zero locus of ϑ. For $i = 1, \ldots, n$ put

$$\dot{u}_i := \dot{\vartheta}\left(z + \sum_{\alpha \in \{1,\ldots,n\}\setminus\{i\}} z(\tilde{P}_\alpha)\right),$$

thereby defining a meromorphic function on $\tilde{X}$ regular away from the lifting to $\tilde{X}$ of the support of the effective divisor $D_i + (n-1)\infty$. If the functions $\dot{u}_i$ differ from one another by constants, then each of the functions $\dot{u}_i$ reduces to a constant and $w_0 = 0$.

Proof. For $i = 1, \ldots, n$, let U_i be the complement of the support of the effective divisor $D_i + (n-1)\infty$. By Proposition 2.6.1 the family $\{U_i\}$ is an open covering of X. Let $\tilde{U}_i \subset \tilde{X}$ be the inverse image of U_i under the covering map. Since $\dot{u}_i$ is regular on $\tilde{U}_i$ and the family $\{\tilde{U}_i\}$ covers $\tilde{X}$, it follows that there exists some holomorphic function $\dot{u}$ on $\tilde{X}$ from which each function $\dot{u}_i$ differs by a constant. Clearly we have

$$\dot{\vartheta}(w + \lambda) = \pi H(\lambda, w_0) + \dot{\vartheta}(w)$$

for all $\lambda \in \Lambda$ and $w \in W$ such that $\vartheta(w) \neq 0$ and hence

$$\sigma^* \dot{u} = \pi H(\sigma, w_0) + \dot{u}$$

for all $\sigma \in \Gamma$. It follows that $\dot{u}$ is the primitive of some holomorphic 1-form on X, say ω. By construction of ω we have

$$\int_{[\tilde{\infty} \to \sigma\tilde{\infty}]} \omega = \pi H(\sigma, w_0)$$

for all $\sigma \in \Gamma$ and hence $\omega = 0$ and $w_0 = 0$ by Proposition 2.3.5. □

3. The edited 4Θ-embedding of a Jacobian

3.1. **The edited 4Θ linear system.**

3.1.1. *The setting.* Fix a compact Riemann surface X of genus $g > 0$ and a universal covering map $\tilde{X} \to X$. Put $\Gamma := \mathrm{Aut}(\tilde{X}/X)$. Fix a basepoint $\infty \in X$ and lifting $\tilde{\infty} \in \tilde{X}$ thereof. Let (W, H, Λ) be the Jacobian of X. Let z be the Abel map $\tilde{X} \to W$ based at $\tilde{\infty}$. Let E be the prime form. Fix a real semicharacter ψ_0 of Λ with respect to H and a theta function ϑ_0 of type (W, H, Λ, ψ_0). More generally, put

$$\vartheta_t(w) = \exp\left(-\pi H(t, w)\right) \vartheta_0(w + t)$$

for all $t \in W$ and correspondingly put

$$\psi_t(\lambda) = \exp(2\pi i \Im H(\lambda, t))\psi_0(\lambda)$$

for all $\lambda \in \Lambda$, so that then ϑ_t is a theta function of type (W, H, Λ, ψ_t). Fix a set

$$0 \in M \subset \frac{1}{2}\Lambda$$

of representatives for the quotient $\frac{1}{2}\Lambda/\Lambda$. Then

$$\{\psi_\mu\}_{\mu \in M}$$

is the family of 4^g real semicharacters of Λ with respect to H.

3.1.2. *Definition.* By the Lefschetz embedding theorem the family of 4^g theta functions

$$\{\vartheta_\mu(2w)\}_{\mu \in M}$$

embeds the quotient W/Λ into projective space. For any nonnegative integer ℓ let $M_{\leq \ell}$ be the subset of M consisting of those μ such that ϑ_μ vanishes at the origin to order $\leq \ell$. We call

$$\{\vartheta_\mu(2w)\}_{\mu \in M_{\leq 1}}$$

the *edited* 4Θ linear system associated to the Jacobian (W, H, Λ). Our goal is to prove the following result.

Theorem 3.1.3. *The edited* 4Θ *linear system embeds* W/Λ *into projective space.*

The theorem can easily be reconciled with its classical-style formulation in the introduction by means of the remarks of §2.2.6. The proof of the theorem requires some preparation and is not going to be completed until the end of §3.3.

3.1.4. *Remark.* As was noted by the referee, it is a part of the folklore of the theory of compact Riemann surfaces that $M_{\leq 1} = M$ generically. Nonetheless our theorem does have some content because in general the inequality $M_{\leq 1} \neq M$ does hold. Indeed, the latter holds "with a vengeance" for hyperelliptic curves of large genus; see [Mumford Tata II, Chap. IIIa, §6, p. 3.105].

3.1.5. *Further remark.* We are grateful to the referee for providing us with a sketch of a proof of the fact that $M_{\leq 1} = M$ generically. We paraphrase the referee's remarks here. On the Teichmüller space T classifying marked Riemann surfaces of genus g each theta characteristic μ defines a subvariety V_μ the points of which correspond to marked Riemann surfaces such that θ_μ vanishes at the origin to order higher than parity requires. What we have to prove is that none of the V_μ equal T. In any case, for at least one even μ and one odd μ we have $V_\mu \neq T$, as can be verified by the study of hyperelliptic curves. But since the mapping class group acts transitively on the possible markings of a Riemann surface, and in turn acts transitively on the even (resp. odd) theta characteristics, we find that $V_\mu \neq T$ for all μ, which is what we wanted to prove.

3.1.6. *Recollection of the Riemann quartic theta identity.* As in Corollary 2.2.8, let $\{C_\mu\}_{\mu\in M}$ be the unique family of constants such that

$$\prod_{i=1}^{4}\vartheta_0\left(w_i\right) = \sum_{\mu\in M} C_\mu \cdot \left\{\begin{array}{cc} & \vartheta_\mu((+w_1+w_2+w_3+w_4)/2) \\ \cdot & \vartheta_\mu((+w_1+w_2-w_3-w_4)/2) \\ \cdot & \vartheta_\mu((+w_1-w_2+w_3-w_4)/2) \\ \cdot & \vartheta_\mu((+w_1-w_2-w_3+w_4)/2) \end{array}\right.$$

for all $w_1, w_2, w_3, w_4 \in W$. Recall that none of the C_μ vanish.

Proposition 3.1.7. *The linear system*

$$\{\vartheta_\mu(2w)\}_{\mu\in M_{\leq 0}}$$

is basepoint free (and hence so is the edited 4Θ *linear system).*

Proof. This is well known. The method of proof in classical terms is likely known to the reader. We sketch the proof here just in order to help orient the reader toward our point of view. Fix $t \in W$ and let w range over W. We have

$$\frac{1}{2}\left[\begin{array}{rrrr} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right]\left[\begin{array}{c} t+w \\ t-w \\ t+w \\ t-w \end{array}\right] = \left[\begin{array}{c} 2t \\ 0 \\ 2w \\ 0 \end{array}\right]$$

and hence

$$\vartheta_0(t+w)^2\vartheta_0(t-w)^2 = \sum_{\mu\in M_{\leq 0}} C_\mu\vartheta_\mu(2t)\vartheta_\mu(0)\vartheta_\mu(2w)\vartheta_\mu(0).$$

The left side of the identity above does not vanish identically as a function of w and hence $\vartheta_\mu(2t) \neq 0$ for some $\mu \in M_{\leq 0}$. □

3.2. **The generating function Z_t.** We now introduce a technical device by means of which we can take maximum advantage of the Riemann quartic theta identity.

3.2.1. *Definition.* To each $t \in W$ we associate a *generating function*

$$Z_t : \tilde{X}^4 \times W \to \mathbb{C}$$

by the rule

$$Z_t(\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{P}_3, w) := \begin{cases} & \vartheta_0(t - z(\tilde{P}_0) - z(\tilde{P}_1) + w) \\ \cdot & \vartheta_0(t + z(\tilde{P}_1) + z(\tilde{P}_3) - w) \\ \cdot & \vartheta_0(t - z(\tilde{P}_2) - z(\tilde{P}_3) + w) \\ \cdot & \vartheta_0(t + z(\tilde{P}_0) + z(\tilde{P}_2) - w). \end{cases}$$

The function Z_t is holomorphic on $\tilde{X}^4 \times W$ and also depends holomorphically on the parameter t. Since

$$\vartheta_0(-w) = \pm\vartheta_0(w), \quad \vartheta_t(w) = \exp(-\pi H(t, w))\vartheta_0(t + w),$$

we can rewrite the definition of Z_t in the form

$$Z_t(\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{P}_3, w) = \begin{cases} & \vartheta_{+t}(w - z(\tilde{P}_0) - z(\tilde{P}_1)) \\ \cdot & \vartheta_{-t}(w - z(\tilde{P}_1) - z(\tilde{P}_3)) \\ \cdot & \vartheta_{+t}(w - z(\tilde{P}_2) - z(\tilde{P}_3)) \\ \cdot & \vartheta_{-t}(w - z(\tilde{P}_0) - z(\tilde{P}_2)). \end{cases}$$

The latter presentation of Z_t is going to be quite convenient later.

3.2.2. *Key properties.* The generating function Z_t viewed as a function of t is a theta function on W relative to Λ. More precisely, we have

$$\begin{aligned} & Z_{t+\lambda}(\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{P}_3, w) \\ = \; & \exp\left(4\pi H(\lambda, t) + 2\pi H(\lambda, \lambda)\right) Z_t(\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{P}_3, w) \end{aligned}$$

for all $\lambda \in \Lambda$. In particular, up to a nonzero constant factor, Z_t depends only on $t \bmod \Lambda$. Further and crucially, we have

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} t - z(\tilde{P}_0) - z(\tilde{P}_1) + w \\ t + z(\tilde{P}_1) + z(\tilde{P}_3) - w \\ t - z(\tilde{P}_2) - z(\tilde{P}_3) + w \\ t + z(\tilde{P}_0) + z(\tilde{P}_2) - w \end{bmatrix}$$

$$= \begin{bmatrix} 2t \\ \\ -z(\tilde{P}_0) + z(\tilde{P}_3) \\ \\ 2w - \sum_{i=0}^{3} z(\tilde{P}_i) \\ \\ -z(\tilde{P}_1) + z(\tilde{P}_2) \end{bmatrix}$$

and hence

$$Z_t(\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{P}_3, w) = \sum_{\mu \in M_{\leq 1}} C_\mu \vartheta_\mu(2t) \cdot \left\{ \begin{array}{c} \vartheta_\mu(-z(\tilde{P}_0) + z(\tilde{P}_3)) \\ \\ \cdot \quad \vartheta_\mu\left(2w - \sum_{i=0}^{3} z(\tilde{P}_i)\right) \\ \\ \cdot \quad \vartheta_\mu(-z(\tilde{P}_1) + z(\tilde{P}_2)) \end{array} \right.$$

by the Riemann quartic theta identity as recollected above combined with Corollary 2.5.7.

3.2.3. *Rationale for the terminology.* We claim that the family

$$\left\{ (\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{P}_3, w) \mapsto \left\{ \begin{array}{c} \vartheta_\mu(-z(\tilde{P}_0) + z(\tilde{P}_3)) \\ \\ \cdot \quad \vartheta_\mu\left(2w - \sum_{i=0}^{3} z(\tilde{P}_i)\right) \\ \\ \cdot \quad \vartheta_\mu(-z(\tilde{P}_1) + z(\tilde{P}_2)) \end{array} \right. \right\}_{\mu \in M_{\leq 1}}$$

of holomorphic functions on $\tilde{X}^4 \times W$ is $\mathbb{C}$-linearly independent. At any rate, by Corollary 2.5.7 there exist points $\tilde{P}_0, \tilde{P}_1$ such that

$$\prod_{\mu \in M_{\leq 1}} \vartheta_\mu(-z(\tilde{P}_0) + z(\tilde{P}_1)) \neq 0,$$

and hence by evaluation at $(\tilde{P}_0, \tilde{P}_0, \tilde{P}_1, \tilde{P}_1) \in \tilde{X}^4$ we can specialize the family in question to a family of functions on W of the form

$$\{D_\mu \vartheta_\mu(2(w + t_0))\}_{\mu \in M_{\leq 1}}$$

where the factors D_μ are nonzero constants and t_0 is some particular point of W. Families of the latter sort are clearly $\mathbb{C}$-linearly independent. Thus the claim is proved. The claim granted, it follows that for any $t \in W$, to know the generating function Z_t up to a nonzero constant factor is to know the image of t under the map to projective space determined by the edited 4Θ linear system, and vice versa. In this sense Z_t packages all the information in the edited 4Θ embedding and hence deserves to be called a generating function.

3.3. **Separation of points and tangent vectors.**

3.3.1. *An* ad hoc *notion of general position.* Suppose we are given an integer $n \geq g$ and a divisor D of X of degree $g - 1$. We say that points $P_0, \ldots, P_{n+1} \in X$ are in (n, D)*-general position* if $P_0, \ldots, P_{n+1}, \infty$ are distinct and

$$h^0\left(D + n\infty - \sum_{i \in I} P_i\right) = 0 = h^0\left(K - D + n\infty - \sum_{i \in I} P_i\right)$$

for all subsets $I \subset \{0, \ldots, n+1\}$ of cardinality n, where as usual K denotes a canonical divisor of X. Now fix $t \in W$ such that $\psi_D = \psi_t$, noting that $\psi_{K-D} = \psi_{-t}$ by Corollary 2.4.6. Also fix distinct points $P_0, \ldots, P_{n+1} \in X \setminus \{\infty\}$ and corresponding liftings $\tilde{P}_0, \ldots, \tilde{P}_{n+1} \in \tilde{X}$. By Corollary 2.5.5 the points $P_0, \ldots, P_{n+1}$ are in (n, D)-general position if and only if

$$\prod_{\substack{I \subset \{0, \ldots, n+1\} \\ \# I = n}} \left(\vartheta_t\left(\sum_{i \in I} z(\tilde{P}_i)\right) \vartheta_{-t}\left(\sum_{i \in I} z(\tilde{P}_i)\right)\right) \neq 0.$$

It follows that the set consisting of $(n+2)$-tuples of points of X in (n, D)-general position is open dense in X^{n+2}.

Proposition 3.3.2. *For all $t, t' \in W$, if we have $Z_t \propto Z_{t'}$, then we have $t \equiv t' \bmod \Lambda$.*

Proof. Fix divisors D and D' of degree $g - 1$ such that $\psi_t = \psi_D$ and $\psi_{t'} = \psi_{D'}$. Fix an integer $n \geq g + 1$. Fix points $P_0, \ldots, P_{n+1} \in X \setminus \infty$ in both (n, D)-general and (n, D')-general position. Fix corresponding liftings $\tilde{P}_0, \ldots, \tilde{P}_{n+1} \in \tilde{X}$. For all indices $i, j = 1, \ldots, n$ we define a function F_{ij} meromorphic on $\tilde{X}$ and regular on $\tilde{X} \setminus \Gamma\tilde{\infty}$ by specializing the generating function Z_t and multiplying by a factor independent of t, as follows:

$$F_{ij}(\tilde{P}) := \begin{cases} & Z_t\left(\tilde{P}, \tilde{P}_i, \tilde{P}_j, \tilde{P}_{n+1}, z(\tilde{P}) + \sum_{\alpha=1}^{n+1} z(\tilde{P}_\alpha)\right) \\ \cdot & \prod_{\alpha\in\{1,\dots,n\}\setminus\{i\}} E(\tilde{P}, \tilde{P}_\alpha) \cdot \prod_{\beta\in\{1,\dots,n\}\setminus\{j\}} E(\tilde{P}, \tilde{P}_\beta). \end{cases}$$

By the second version of the definition of the generating function Z_t we have

$$F_{ij}(\tilde{P}) = \begin{cases} & \vartheta_{+t}\left(\sum_{\alpha\in\{1,\dots,n+1\}\setminus\{i\}} z(\tilde{P}_\alpha)\right) \\ \cdot & \vartheta_{-t}\left(z(\tilde{P}) + \sum_{\alpha\in\{1,\dots,n\}\setminus\{i\}} z(\tilde{P}_\alpha)\right) \cdot \prod_{\alpha\in\{1,\dots,n\}\setminus\{i\}} E(\tilde{P}, \tilde{P}_\alpha) \\ \cdot & \vartheta_{+t}\left(z(\tilde{P}) + \sum_{\beta\in\{1,\dots,n\}\setminus\{j\}} z(\tilde{P}_\beta)\right) \cdot \prod_{\beta\in\{1,\dots,n\}\setminus\{j\}} E(\tilde{P}, \tilde{P}_\beta) \\ \cdot & \vartheta_{-t}\left(\sum_{\beta\in\{1,\dots,n+1\}\setminus\{j\}} z(\tilde{P}_\beta)\right). \end{cases}$$

The latter presentation of F_{ij} makes it clear that F_{ij} does not vanish identically. Indeed, since the points $P_0, \dots, P_{n+1}$ are in (n, D)-general position, we have

$$\prod_{i=1}^{n}\prod_{j=1}^{n} F_{ij}(\tilde{P}_0) \neq 0.$$

It is easily verified that F_{ij} is a theta function on $\tilde{X}$ relative to Γ and hence represents some divisor on X, say D_{ij}. By Proposition 2.6.1 the divisor

$$D^* := (n-1)\infty + \min_{i=1}^{n} D_{i1}$$

belongs to the divisor class of $D + \infty$. Since $Z_t \propto Z_{t'}$, a repetition of the preceding argument proves that D^* also belongs to the divisor class of $D' + \infty$. Therefore the divisors D and D' are linearly equivalent, hence $\psi_t = \psi_D = \psi_{D'} = \psi_{t'}$ and hence $t \equiv t' \bmod \Lambda$. $\square$

Proposition 3.3.3. *For all $t, w_0 \in W$, if the meromorphic function*

$$\left.\frac{\partial}{\partial s}\log Z_{t+sw_0}\right|_{s=0}$$

on $\tilde{X}^4 \times W$ reduces to a constant, then we have $w_0 = 0$.

Proof. Put

$$\dot{\vartheta}_{\pm t}(w) := \left.\frac{\partial}{\partial s}\log \vartheta_{\pm t}(w + sw_0)\right|_{s=0},$$

thereby defining a meromorphic function on W regular away from the zero locus of $\vartheta_{\pm t}$. Note that

$$\vartheta_{\pm(t+sw_0)}(w) = \exp(-\pi H(\pm w_0, w)\bar{s} + \pi H(t, w_0)s)\vartheta_{\pm t}(w \pm sw_0)$$

and hence

$$\left.\frac{\partial}{\partial s}\log \vartheta_{\pm(t+sw_0)}\right|_{s=0} = \pi H(t, w_0) \pm \dot{\vartheta}_{\pm t}.$$

Now fix a divisor D of X of degree $g-1$ such that $\psi_t = \psi_D$, an integer $n \geq g+1$, points $P_0, \dots, P_{n+1} \in X \setminus \infty$ in (n, D)-general position and corresponding liftings $\tilde{P}_0, \dots, \tilde{P}_{n+1} \in \tilde{X}$. For $i, j = 1, \dots, n$ we have

$$\begin{aligned}
\text{constant} &= \left.\frac{\partial}{\partial s}\log Z_{t+sw_0}\left(\tilde{P}_0, \tilde{P}_i, \tilde{P}_j, \tilde{P}_{n+1}, z + \sum_{\alpha=1}^{n+1} z(\tilde{P}_\alpha)\right)\right|_{s=0} \\
&= 4\pi H(t, w_0) + \left\{\begin{array}{rl}
& \dot{\vartheta}_{+t}\left(\displaystyle\sum_{\alpha\in\{1,\dots,n+1\}\setminus\{i\}} z(\tilde{P}_\alpha)\right) \\
- & \dot{\vartheta}_{-t}\left(z + \displaystyle\sum_{\alpha\in\{1,\dots,n\}\setminus\{i\}} z(\tilde{P}_\alpha)\right) \\
+ & \dot{\vartheta}_{+t}\left(z + \displaystyle\sum_{\beta\in\{1,\dots,n\}\setminus\{j\}} z(\tilde{P}_\beta)\right) \\
- & \dot{\vartheta}_{-t}\left(\displaystyle\sum_{\beta\in\{1,\dots,n+1\}\setminus\{j\}} z(\tilde{P}_\beta)\right).
\end{array}\right.
\end{aligned}$$

We conclude that $w_0 = 0$ by Proposition 2.6.2. ☐

3.3.4. *Completion of the proof of Theorem 3.1.3.* By Proposition 3.1.7 we know at least that the edited 4Θ linear system defines a mapping from W/Λ to projective space. But then, on account of the generating function interpretation of Z_t provided in §3.2.3, we know that this mapping separates points by Proposition 3.3.2 and separates tangent vectors by Proposition 3.3.3. $\square$

3.4. **"Tying together algebraic and analytic Jacobians".** The turn of phrase here is borrowed from [Mumford Tata II].

3.4.1. *Identifications.* Continuing in the setting of the proof of Theorem 3.1.3, we now also contemplate the situation discussed in the first couple of paragraphs of the introduction. Recall that in the setting of the introduction we make the yet stronger assumption that

$$n \geq g + 2.$$

Write

$$\psi_0 = \psi_{D_0}$$

for some half-canonical divisor D_0. Write

$$\mathcal{E} = \mathcal{O}_X(D_0 + S + n\infty), \quad \mathcal{T} = \mathcal{O}_X(T)$$

where S and T are divisors of degree zero. Select $s, t \in W$ such that

$$\psi_s = \psi_{D_0+S}, \quad \psi_{D_0+T} = \psi_t.$$

Make the evident identifications

$$H^0\left(X, \mathcal{T}^{\pm 1} \otimes \mathcal{E}\right) = \left(\begin{array}{l} \text{space of theta functions} \\ \text{of type } (X, \infty, \psi_{s\pm t}, n) \end{array}\right),$$

thereby inducing an identification of each matrix entry $\mathrm{abel}(\mathcal{T})_{ij}$ with a function on $\tilde{X}^{\{0,\ldots,n+1\}}$. We are going to write down a formula for $\mathrm{abel}(\mathcal{T})_{ij}(\tilde{P}_0, \ldots, \tilde{P}_{n+1})$ in terms of the generating function Z_t and some other factors independent of t. Since knowledge of Z_t up to a nonzero constant multiple and knowledge of the image of t under the map to projective space defined by the edited 4Θ linear system are equivalent, the formula we are going to write down "is" the promised factorization of the map $\mathcal{T} \mapsto \mathrm{abel}(\mathcal{T})$ through the edited 4Θ embedding.

3.4.2. *Plugging into the determinant identity.* As in the introduction, let u (resp., v) be a row vector of length n with entries forming a basis of $H^0(X, \mathcal{T}^{-1} \otimes \mathcal{E})$ (resp., $H^0(X, \mathcal{T} \otimes \mathcal{E})$) over the ground field, the latter now taken to be $\mathbb{C}$. Taking into account the identifications made above, and after adjusting u and v by suitably chosen nonzero constant factors, we have

$$\left(\det_{i,j=1}^{n} u_j^{(i)}\right)\left(\tilde{P}_1,\dots,\tilde{P}_n\right) = \left\{\begin{array}{ll} & \vartheta_{-t}\left(s+\sum_{i=1}^{n} z(\tilde{P}_i)\right) \\ \cdot & \exp\left(-\pi H\left(s, \sum_{i=1}^{n} z(\tilde{P}_i)\right)\right) \\ \cdot & \prod_{1\leq i<j\leq n} E(\tilde{P}_i,\tilde{P}_j) \end{array}\right.$$

and

$$\left(\det_{i,j=1}^{n} v_j^{(i)}\right)\left(\tilde{P}_1,\dots,\tilde{P}_n\right) = \left\{\begin{array}{ll} & \vartheta_{t}\left(s+\sum_{i=1}^{n} z(\tilde{P}_i)\right) \\ \cdot & \exp\left(-\pi H\left(s, \sum_{i=1}^{n} z(\tilde{P}_i)\right)\right) \\ \cdot & \prod_{1\leq i<j\leq n} E(\tilde{P}_i,\tilde{P}_j) \end{array}\right.$$

by plugging into the determinant identity of Proposition 2.5.4, it being noted that

$$\exp(-\pi H(s,w))\vartheta_{\pm t}(s+w)$$

is a theta function of type $(W, H, \Lambda, \psi_{s\pm t})$.

3.4.3. *Completion of the calculation.* Finally, we have

$$
\begin{aligned}
&\mathrm{abel}(\mathcal{T})_{ij}(\tilde{P}_0,\ldots,\tilde{P}_{n+1})\\
= \quad & Z_t\left(\tilde{P}_0,\tilde{P}_i,\tilde{P}_j,\tilde{P}_{n+1},s+\sum_{\alpha=0}^{n+1} z(\tilde{P}_\alpha)\right)\\
\cdot \quad & \exp\left(-\pi H\left(s,-2(z(\tilde{P}_0)+z(\tilde{P}_i)+z(\tilde{P}_j)+z(\tilde{P}_{n+1}))+4\sum_{\alpha=0}^{n+1} z(\tilde{P}_\alpha)\right)\right)\\
\cdot \quad & \prod_{\substack{\alpha,\beta\in\{0,\ldots,n+1\}\setminus\{0,i\}\\ \alpha<\beta}} E(\tilde{P}_\alpha,\tilde{P}_\beta)\cdot \prod_{\substack{\alpha,\beta\in\{0,\ldots,n+1\}\setminus\{i,n+1\}\\ \alpha<\beta}} E(\tilde{P}_\alpha,\tilde{P}_\beta)\\
\cdot \quad & \prod_{\substack{\alpha,\beta\in\{0,\ldots,n+1\}\setminus\{j,n+1\}\\ \alpha<\beta}} E(\tilde{P}_\alpha,\tilde{P}_\beta)\cdot \prod_{\substack{\alpha,\beta\in\{0,\ldots,n+1\}\setminus\{0,j\}\\ \alpha<\beta}} E(\tilde{P}_\alpha,\tilde{P}_\beta).
\end{aligned}
$$

Note that among the factors on the right side of the identity, only the very first depends on t. Note also that since $n \geq g+2$, the formula above determines the generating function Z_t uniquely in terms of, say, the matrix entry $\mathrm{abel}(\mathcal{T})_{12}$. The formula above generalizes the genus one identity [Anderson 2002, (49)] to arbitrary genus.

References

[Anderson 2002] Anderson, G. W.: *Abeliants and their application to an elementary construction of Jacobians*, Adv. in Math. **172**(2002), 169-205.

[Fay] Fay, J.: *Theta functions on Riemann surfaces.* Springer Lecture Notes **352**, Berlin New York 1973.

[Griffiths-Harris] Griffith, P.; Harris, J.: *Principles of Algebraic Geometry.* John Wiley and Sons, New York 1978

[Mumford Tata I] Mumford, D.: *Tata Lectures on Theta I.* Progr. Math. **28**(1983), Birkhäuser Boston

[Mumford Tata II] Mumford, D.: *Tata Lectures on Theta II.* Progr. Math. **43**(1984), Birkhäuser Boston

[Mumford Tata III] Mumford, D.: *Tata Lectures on Theta III.* Progr. Math. **97**(1991), Birkhäuser Boston

[Segal-Wilson 1985] Segal, G.; Wilson, G.: *Loop groups and equations of KdV type.* Publ. IHES **61**(1985)5-65.

[Weil VK] Weil, A.: *Introduction à l'étude des variétés kählériennes.* Hermann, Paris 1971

University of Minnesota, MN 55455 USA

E-mail address: `gwanders@math.umn.edu`