HOMEWORK FOR MATH 8590

Homework 1, due on Oct 21 in class

Problem 1. Fix a smooth function $b(y)$ on $[0,1]$ with $b'(y) > 0$ for $y \in [0,1]$. Assume that a sequence of functions $\psi_n \in H_0^1(0,1)$ satisfy

$$\lim_{n\to\infty} \|\psi_n - \psi\|_{H^1(0,1)} = 0,$$

for some $\psi \in H_0^1(0,1)$. Assume that $\epsilon_n > 0, \mu_n \in \mathbb{R}$ and $\lim_{n\to 0} \epsilon_n = 0$, $\lim_{n\to\infty} \mu_n = b(1/2)$. Calculate the limit

$$\lim_{n\to\infty} \frac{\psi_n(y)}{b(y) - \mu_n + i\epsilon_n}$$

in the sense of distributions.

Problem 2. Fix a smooth function $b(y)$ on $[0,1]$ with $b'(y) > 0$ for $y \in [0,1]$. Assume in addition that $b''(y)$ vanishes only at $y = 1/2$, that

$$K(y) := \frac{b''(y)}{b(y) - b(1/2)} < 0, \qquad \text{for } y \in [0,1]$$

and that

$$-\frac{d^2}{dy^2} + K(y)$$

has at least one negative eigenvalue $-\alpha_0^2 < 0$. Consider the equation

$$(b(y) - \lambda)(\alpha^2 - \frac{d^2}{dy^2})\psi(y) + b''(y)\psi = 0, \qquad \text{for } y \in (0,1) \tag{1}$$

with parameters $\alpha \in (0, \alpha_0^2)$, $\lambda \in \mathbb{C}$ with imaginary part of $\lambda > 0$, and $\psi \in H^2 \cap H_0^1(0,1)$. Show that (1) has nontrivial solutions λ, ψ, if $\alpha = \sqrt{\alpha_0^2 - \epsilon}$ with sufficiently small $\epsilon > 0$. Moreover, give an example of b satisfying the assumptions.

Problem 3. (Optional, some research required) Suppose that b is given as in problem 2, and that

$$-\frac{d^2}{dy^2} + K(y)$$

has $m \geq 1$ negative eigenvalues which are < -1. Is it true that the equation

$$(b(y) - \lambda)(1 - \frac{d^2}{dy^2})\psi(y) + b''(y)\psi = 0, \qquad \text{for } y \in (0,1) \tag{2}$$

has at least m nontrivial solutions λ_i with imaginary part of $\lambda > 0$, $\psi \in H^2 \cap H_0^1(0,1), \psi_i \not\equiv 0, i \in [1,m]$ with $\lambda_i \neq \lambda_j$ for $i \neq j$?

Problem 4. Fix an interval $I \subseteq \mathbb{R}$. Let $f(\alpha), \alpha \in I$ be a one parameter family of distributions on $\mathbb{R}$, i.e., for each α in some interval I, $f(\alpha)$ is a generalized function on $\mathbb{R}$. Define the integral

$$\int_I f(\alpha)\, d\alpha$$

of $f(\alpha)$ as the distribution

$$\left[\int_I f(\alpha)\, d\alpha\right] \phi := \int_I [f(\alpha)]\, \phi\, d\alpha,$$

for any $\phi \in C_0^\infty(\mathbb{R})$. Calculate the distributions

$$\int_0^\infty e^{i\alpha x}\, d\alpha, \qquad \text{and} \qquad \int_{\mathbb{R}} e^{i\alpha x}\, d\alpha.$$