① Lecture # 1. Fourier transforms and distributions

References: Appendix B: Theory of distributions, Function Analysis, Peter Lax,

For $f \in L^1(\mathbb{R}^n)$, we can define

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx.$$

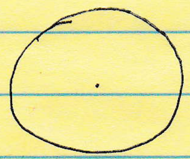
Claim: $\boxed{\text{Then } \hat{f} \in C_0(\mathbb{R}^n).}$

$C_0(\mathbb{R}^n)$ is the space of continuous functions on $\mathbb{R}^n$ that vanish at ∞.

To prove the claim, we need to use the fact C_c^∞ functions are dense in $L^1(\mathbb{R}^n)$. Suppose this is the case. We can take $\varphi_n \in C_c^\infty$ ... smooth and compactly supported, so that $\varphi_n \to f$ in $L^1(\mathbb{R}^n)$.

Then $\hat{\varphi}_n \to \hat{f}$ in $C(\mathbb{R}^n)$. Since $\hat{\varphi}_n \in C_0(\mathbb{R}^n)$, we conclude that $\hat{f} \in C_0(\mathbb{R}^n)$.

Smooth function with compact support:



$$\eta(x) := \begin{cases} C\, e^{\frac{1}{|x|^2-1}} & \text{if } |x|<1 \\ 0 & \text{if } |x|\geq 1 \end{cases}$$

② $f_\epsilon(x) = \int f(y)\, \epsilon^{-n} \eta\left(\frac{x-y}{\epsilon}\right) dy$ is smooth, if $f \in L^1$

and $f_\epsilon \to f$.

Let us now introduce the space of Schwarz functions.

$S(\mathbb{R}^n)$ is the space of functions f s.t for any $M > 0$,

$$\sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \le M \\ \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n}} \langle x \rangle^M |\partial^\alpha f(x)| < \infty.$$

$\langle x \rangle := \sqrt{|x|^2 + 1}$.

E.g. $P(x) e^{-|x|^2}$, P -- polynomial.

<u>Lemma</u> If $f \in S(\mathbb{R}^n)$, then $\hat{f} \in S(\mathbb{R}^n)$.

proof:

$$\partial^\alpha \hat{f}(\xi) = C_\alpha \int x^\alpha f(x) e^{-ix\cdot\xi} dx$$

$$\xi^\beta \partial^\alpha \hat{f}(\xi) = C_\alpha \int \xi^\beta x^\alpha f(x) e^{-ix\cdot\xi} dx$$

$$= C_{\alpha,\beta} \int x^\alpha f(x)\, \partial^\beta e^{-ix\cdot\xi} dx$$

$$= \tilde{C}_{\alpha\beta} \int \partial^\beta (x^\alpha f(x))\, e^{-ix\cdot\xi} dx$$

By the fact that $f \in S(\mathbb{R}^n)$, we conclude that

$$\sup_{\xi \in \mathbb{R}^n} |\xi^\beta \partial^\alpha \hat{f}(\xi)| < +\infty. \qquad \square$$

<u>Proposition</u> (properties of Fourier transform).

③

(1) $\widehat{\partial_{x_j} f} = i\xi_j \hat{f}(\xi)$.

(2) $\widehat{x_j f} = i\partial_{\xi_j} \hat{f}$

(3) $\widehat{f(\cdot - b)} = \cancel{e^{-bx}}\, e^{-ib\cdot\xi} \hat{f}$

(4) $\widehat{e^{-iax} f}(\xi) = \hat{f}(\xi + a)$.

Inversion of Fourier transform

Take a smooth function Φ s.t. $\Phi \equiv 1$ in B_1

and $\operatorname{Supp} \Phi \subseteq B_2$.

(Such a function can be obtain by setting $\Phi = \chi_{B_{4/3}} * \eta_{\frac{1}{10}}$

where $\chi_{B_{4/3}}$ is the characteristic function of $B_{4/3}$, η is the explicit function constructed earlier. $\eta_{\frac{1}{10}}(x) = \cancel{10\eta(\frac{x}{10})}\; \eta(10x)$).

Let us take $f \in S(\mathbb{R}^n)$, $\hat{f} \in S(\mathbb{R}^n)$.

We evaluate $g(x) := \int \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi$

$\forall\, \varepsilon > 0$, let us consider

$$g_\epsilon(x) := \int \hat{f}(\xi)\, \Phi(\varepsilon\xi)\, e^{ix\cdot\xi}\, d\xi$$

$$= \int \int f(y) e^{-iy\cdot\xi}\, \Phi(\varepsilon\xi)\, e^{ix\cdot\xi}\, dx\, d\xi$$

$$= \int f(y) \int \Phi(\varepsilon\xi)\, e^{i(x-y)\xi}\, d\xi\, dx$$

$$= \varepsilon^{-n} \int f(y) \int \Phi(\xi)\, e^{i\varepsilon^{-1}(x-y)\xi}\, d\xi\, dx$$

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$$= \int f(y) \frac{1}{\varepsilon^n} \hat{\Phi}\left(\frac{y-x}{\varepsilon}\right) dy$$

$$= \int f(x+\varepsilon y) \hat{\Phi}(y)\, dy$$

Hence $g(x) = \lim_{\varepsilon\to 0} g_\varepsilon(x) = f(x) \int \hat{\Phi}(y)\, dy$

$$= C_0 f(x).$$

Up to a constant, the inverse Fourier transform "inverts" the Fourier transform.

To determine C_0. let us consider

$f(x) = e^{-|x|^2/2}$. $\quad \hat{f}(\xi) = (2\pi)^{n/2} e^{-|\xi|^2/2}$.

$$\int \hat{f}(\xi) e^{ix\cdot\xi} dx = (2\pi)^n e^{-|x|^2/2}.$$

Thus $\check{g}(x) = Tg(x) := \frac{1}{(2\pi)^n} \int g(\xi) e^{ix\cdot\xi} d\xi$

inverts Fourier transfm.

$$\frac{1}{(2\pi)^n} \int \hat{f}(\xi) e^{ix\cdot\xi} d\xi = f(x).$$

<u>Parseval's identity</u>

We first note the algebraic identity: $f, g \in S(\mathbb{R}^n)$,

$\widehat{f*g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$, where

$$f*g(x) = \int f(x-y) g(y)\, dy.$$

Note also that $\widehat{f(-x)}(\xi) = \overline{\hat{f}(\xi)}$

Hence, $|\hat{f}(\xi)|^2 = \cancel{f * \bar{f}(-\cdot)}\ \widehat{f*g}(\xi)$

with $g(x) = \bar{f}(-x)$.

Then $\int |\hat{f}(\xi)|^2 \, d\xi = \int \widehat{f*g}(\xi)\, d\xi$

$$= \int \widehat{f*g}(\xi) e^{i0\cdot\xi} d\xi = (2\pi)^n\, f*g(0)$$

$$= (2\pi)^n \cancel{\int f(x)} \int f(0-y)\, g(y)\, dy$$

$$= (2\pi)^n \int f(-y)\bar{f}(-y)\, dy = (2\pi)^n \int |f|^2$$

~~This~~ That is, $\|\hat{f}\|_{L^2} = (2\pi)^{\frac{n}{2}} \|f\|_{L^2}$.

By approximation, we can thus define Fourier transform of L^2 functions.

However, in applications to PDEs, we often need to take Fourier transform for functions which are more general that Schwarz or L^2/L^1 functions. Below we introduce a wider function space in which Fourier transform can be naturally defined.

Tempered distributions

Tempered distributions are ~~fu~~ linear functions from $S(\mathbb{R}^n) \to \mathbb{R}$. $T: S(\mathbb{R}^n) \to \mathbb{R}$.

We also require T to be bounded in the following sense.

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$\exists\ M > 1$ s.t

$$|T(f)| \leq C \left\| \langle x \rangle^M \sum_{|\alpha| \leq M} |\partial^\alpha f(x)| \right\|_{L^\infty}.$$

The class of tempered distributions are quite large.

e.g. if $\langle x \rangle^{-M} f \in L^1(\mathbb{R}^n)$ then

f can be identified with a tempered distribution, in the following sense:

$$\cancel{T(f)} \quad T(g) = \int f(x)\, g(x)\, dx, \quad \forall g \in S(\mathbb{R}^n).$$

Examples of tempered distributions:

(1) Dirac mass. $\delta(x_0)$.

(2) P.V. $\frac{1}{x}$, $x \in \mathbb{R}$

(3) L^p functions, $p \geq 1$.

(4) ~~Measures~~ Compactly supported measures μ.

~~Examples of fu~~

Other Counter examples: $Tg = \sum_{j=1}^{\infty} 2^{-j} \frac{d^j}{dx^j} g(0).$

$$Tg = \sum_{j=1}^{\infty} e^j g(j).$$

$\forall f \in S'(\mathbb{R}^n)$, we can define $\partial^\alpha f : S(\mathbb{R}^n) \to \mathbb{R}$ as

$$\partial^\alpha f(g) = (-1)^{|\alpha|} f(\partial^\alpha g).$$

~~Similarly~~, Similarly, for a function m, $mf(g) := f(mg)$.

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Lemma. Let $f \in S'(\mathbb{R}^n)$ with $\operatorname{Supp} f = \{0\}$, in the sense that $f(g) = 0$ for any $g \in S(\mathbb{R}^n)$ with $0 \notin \operatorname{Supp} g$. Then $f = \sum_{|\alpha| \leq M} A_\alpha \partial^\alpha \delta_0$.

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proof: Since f is a tempered distribution, $\exists K \geq 1$,

s.t $\quad |f(g)| \leq C \sup_{|\alpha| \leq K} \langle x \rangle^K |\partial^\alpha g(x)|, \quad \forall g \in S(\mathbb{R}^n)$.

We claim that $\forall g \in S(\mathbb{R}^n)$, if $\partial^\alpha g(0) = 0 \quad \forall |\alpha| \leq K+10$, then $\quad f(g) = 0$.

~~Since~~ ~~Supp~~ It is clear that

$$|f(g)| = |f(\Psi(\tfrac{x}{\varepsilon}) g)| \leq C \sup_{\substack{|\alpha| \leq K \\ x \in B_\varepsilon}} \varepsilon^{-K} |\partial^\alpha g(x)|$$

$$\leq C \varepsilon^{10} \to 0.$$

Thus $\quad \forall g \in S(\mathbb{R}^n)$,

$$f(g) = f\Big(\sum_{|\alpha| \leq K+10} \frac{\partial^\alpha g(0) x^\alpha}{|\alpha|!} \Psi(x) \Big)$$

$$= \sum_{|\alpha| \leq K+10} \frac{\partial^\alpha g(0)}{|\alpha|!} f(x^\alpha \Psi(x)).$$

where $\Psi(x) \equiv 1$ on B_1, $\Psi \in C_c^\infty$. $\square$

Next, we define Fourier transform for tempered distributions, by duality argument.

(8) We use the observation that $\forall f, g \in S(\mathbb{R}^n)$

$$\int \hat{f}(\xi)\, g(\xi)\, d\xi = \iint f(x)\, e^{-ix\cdot\xi}\, g(\xi)\, dx\, d\xi = \int f(x)\, \hat{g}(x)\, dx$$

it is natural to define $\forall f \in S'(\mathbb{R}^n)$, $\hat{f}$ as a linear function $S(\mathbb{R}^n) \to \mathbb{R}$ by

$$\hat{f}(g) = f(\hat{g})$$

<u>Lemma</u> $\forall f \in S'(\mathbb{R}^n) \Rightarrow \hat{f} \in S'(\mathbb{R}^n)$.

proof: Since $f \in S'(\mathbb{R}^n)$, $\exists M > 1$, s.t.

$$|f(g)| \le C \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \le M}} \langle x\rangle^M |\partial^\alpha g(x)|.$$

$$\hat{f}(g) = f(\hat{g}).$$

$$|\hat{f}(g)| \le C \sup_{\substack{\xi \in \mathbb{R}^n \\ |\alpha| \le M}} \langle \xi\rangle^M |\partial^\alpha \hat{g}(\xi)| \le C \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \le M'}} \langle x\rangle^{M'} |\partial^\alpha g(x)|.$$

$\square$

Thus the Fourier transform is a very natural generalization from $S(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$. In practice, to calculate the Fourier transform for $f \in S'(\mathbb{R}^n)$, it is often useful to use nice functions to approximate f. The following Lemma justifies such approximations.

Lemma. Suppose $f_n \in S'(\mathbb{R}^n)$ satisfy the uniform bddness:

$$\forall g \in S(\mathbb{R}^n), \quad |f_n(g)| \le C \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \le M}} \langle x\rangle^M |\partial^\alpha f(x)|,$$

and $f_n \to f$ in the sense that $f_n(g) \to f(g)$ for any $g \in S(\mathbb{R}^n)$

(9) Then $\hat{f_n}(\xi) \to \hat{f}(\xi)$ in $S'(\mathbb{R}^n)$.

proof: As the last proof ~~suggest~~ shows, f_n are uniformly bdd in $S'(\mathbb{R}^n)$.

$\forall g \in S(\mathbb{R}^n)$, $\hat{f_n}(g) = f_n(\hat{g}) \to f(\hat{g})$. □.

~~Lemma Suppose that $f_n \in L^1_{loc}(\mathbb{R}^n)$ and $f_n \in S'(\mathbb{R}^n)$, $f_n \to f$ in $S'(\mathbb{R}^n)$ for some $f \in S'(\mathbb{R}^n)$. Then $f \in L^1_{loc}(\mathbb{R}^n)$ M~~

Examples (0). $f \in S'(\mathbb{R}^n)$, then $\widehat{\partial_j f} = i\xi_j \hat{f}(\xi)$; $\widehat{x_j f} = i\partial_{\xi_j}\hat{f}$;
$\widehat{f(\cdot - b)} = e^{-ib\cdot\xi}\hat{f}$; $\widehat{e^{-iax}f}(\xi) = \hat{f}(\xi + a)$.

(1) $f(x) = \frac{1}{|x|^2}$ $\mathbb{R}^n$, $n \geq 3$.

First observation $\hat{f}$ is radial. ($\exists$ pointwisely def. function)

Second observation: Notice the scaling symmetry

$f(\lambda x) = \lambda^{-2} f(x)$.

$$\left.\begin{aligned} \widehat{f(\lambda\cdot)} &= \lambda^{-n}\hat{f}\left(\tfrac{\cdot}{\lambda}\right) \\ \widehat{\lambda^{-2}f} &= \lambda^{-2}\hat{f}(\cdot) \end{aligned}\right\} \Rightarrow \hat{f}(\cdot) = \lambda^{-n+2}\hat{f}\left(\tfrac{\cdot}{\lambda}\right)$$

$\Rightarrow \hat{f}$ is $-n+2$ homogeneous.

Two observations together imply $\hat{f}(\xi) = C_n |\xi|^{-n+2}$.

(1) Suppose $u(x) = \widehat{\frac{1}{|\xi|^2}}(x)$, then

$$-\Delta u = \hat{1}(\xi) = (2\pi)^n \delta_0 .$$ integrate over B_1,

We get $$C_n \int_{\partial B_1} \partial_r r^{-n+2} \, d\sigma = (2\pi)^n ,$$ from this one can

calculate $$C_n = \frac{(2\pi)^n}{n-2 \, |\partial B_1|} .$$ $\square$

$\boxed{(2)}$ Calculate $$u(x,t) = \cancel{e^{-|\xi|^2 t}} \left(e^{-|\xi|^2 t}\right)^{\vee}(x)$$

$$= t^{-n} \left(e^{-|\xi|^2}\right)^{\vee}\left(\frac{x}{t}\right)$$

It suffices to calculate $\left(e^{-|\xi|^2}\right)^{\vee}(x)$

$$\left(e^{-|\xi|^2}\right)^{\vee}(x) = \frac{1}{(2\pi)^n} \int e^{-|\xi|^2} e^{ix\cdot\xi} \, d\xi = \frac{1}{(2\pi)^n} \prod_{j=1}^{n} \int e^{-\xi_j^2 + i x_j \xi_j} \, d\xi_j$$

$$= \frac{1}{(2\pi)^n} \prod_{j=1}^{n} e^{-\frac{x_j^2}{4}} \cdot \left(\int e^{-b^2} \, db\right)^n = \frac{1}{\pi^{n/2}} e^{-\frac{|x|^2}{4}}$$

Hence $$u(x,t) = c_n \cdot t^{-n} \, e^{-\frac{|x|^2}{4t}} .$$

$\boxed{(3)}$ $$u(x,t) = \cancel{\left(e^{i\Delta t}\right)} \left(e^{-i|\xi|^2 t}\right)^{\vee}(x)$$

$$= t^{-n} u(x,1)$$

$$u(x,1) = \left(e^{-i|\xi|^2}\right)^{\vee}(x) .$$

We can use approximating, and calculate for $\varepsilon > 0$,

$$\frac{1}{(2\pi)^n} \int e^{-\varepsilon|\xi|^2 - i|\xi|^2} e^{ix\cdot\xi} \, d\xi = \frac{1}{(2\pi)^n} \prod_{j=1}^{n} \int e^{-\epsilon \xi_j^2 - i\xi_j^2 + i x_j \xi_j} \, d\xi_j$$

$$= \frac{1}{(2\pi)^n} \prod_{j=1}^{n} \int e^{-\left[(\varepsilon+i)\xi_j^2 - i x_j \xi_j + \frac{1}{4}\left(\frac{i}{\sqrt{\varepsilon+i}} x_j\right)^2\right]} d\xi_j \; \times \; e^{-\frac{|x|^2}{4(\varepsilon+i)}} .$$

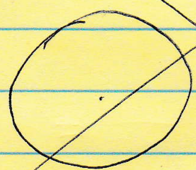
⑪ $= \frac{1}{(2\pi)^n}\left[\prod_{j=1}^{n} \int e^{-\left(\sqrt{\varepsilon+i}\,\xi_j - \frac{i}{\sqrt{\varepsilon+i}} x_j\right)^2} d\xi_j\right] \times e^{-\frac{|x|^2}{4(\varepsilon+i)}}$

$\cancel{= \frac{1}{(2\pi)^n}\prod_{j=1}^{n}\int}$ $= \frac{1}{(2\pi)^n} e^{-\frac{|x|^2}{4(\varepsilon+i)}} \frac{1}{\sqrt{\varepsilon+i}} \prod_{j=1}^{n} \int e^{-(\sqrt{\varepsilon+i}\,\xi_j)^2} d\xi_j \sqrt{\varepsilon+i}$

(Change of Contour of integration) $= \frac{1}{(2\pi)^n} e^{-\frac{|x|^2}{4(\varepsilon+i)}} \prod_{j=1}^{n} \int e^{-\xi_j^2} d\xi_j$

$= C_n e^{-|x|^2/4(\varepsilon+i)}$. $\varepsilon \to 0$. $u(x,t) = C_n t^{-n} e^{-\frac{|x|^2}{4it}}$.

(4). Fourier integral of Surface measure of S^{n-1}.



$\cancel{\hat{f}\ \hat{f}(x}$ $\cancel{\xi = |\xi|\omega,\ \omega\in S}$

$\hat{f}(\xi) = \int_{\partial B_1} e^{-ix\cdot\xi}\, d\sigma(x)$

$= \int_{\partial B_1} e^{-i|x|}$. $= \int_{\partial B_1} e^{-ix\cdot\omega|\xi|}$

Clearly $\hat{f}(\xi)$ is a radial function. It suffices to calculate $\hat{f}(Ae_1) = c_n \int_{\partial B_1} e^{-iAx_1} d\sigma(x)$

$= C_n \int_{-1}^{1} dx_1\, e^{-iAx_1}$.

(4) Fourier transform of $\delta \otimes \frac{\sin(|\xi|t)}{|\xi|}$.

$u(x,t) = \left(\frac{\sin(|\xi|t)}{|\xi|}\right)^\vee(x). = t^{-n+1}\left(\frac{\sin|\xi|}{|\xi|}\right)^\vee\left(\frac{x}{t}\right).$

It suffices to calculate $\left(\frac{\sin|\xi|}{|\xi|}\right)^\vee$.