Mathematics of Image and Data Analysis
Math 5467

Lecture 11: Parseval's identities and Convolution

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Last time

- The Fast Fourier Transform (FFT)

Today

- Parseval's identities
- Convolution and the DFT

Recall

Definition 1. The *Discrete Fourier Transform (DFT)* is the mapping $\mathcal{D} : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

$$\mathcal{D}f(\ell) = \sum_{k=0}^{n-1} f(k)\omega^{-k\ell} = \sum_{k=0}^{n-1} f(k)e^{-2\pi ik\ell/n},$$

where $\omega = e^{2\pi i/n}$ and $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ is the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n$.

The DFT can be viewed as a change of basis into the orthogonal basis functions

$$u_\ell(k) = \omega^{k\ell} = e^{2\pi ik\ell/n}$$

for $\ell = 0, 1, \ldots, n-1$.

Inverse Fourier Transform

Theorem 2 (Fourier Inversion Theorem)**.** *For any* $f \in L^2(\mathbb{Z}_n)$ *we have*

$$f(k) = \frac{1}{n}\sum_{\ell=0}^{n-1} \mathcal{D}f(\ell)\omega^{k\ell} = \frac{1}{n}\sum_{\ell=0}^{n-1} \mathcal{D}f(\ell)e^{2\pi i k\ell/n}. \tag{1}$$

Definition 3 (Inverse Discrete Fourier Transform)**.** The *Inverse Discrete Fourier Transform (IDFT)* is the mapping $\mathcal{D}^{-1} : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

$$\mathcal{D}^{-1}f(\ell) = \frac{1}{n}\sum_{k=0}^{n-1} f(k)\omega^{k\ell} = \frac{1}{n}\sum_{k=0}^{n-1} f(k)e^{2\pi i k\ell/n}.$$

Adjoint of $\mathcal{D}$

$$\langle f, g \rangle = \sum_{k=0}^{n-1} f(k)\overline{g(k)}$$

Hermetian inner product

We first show that $\mathcal{D}^{-1}$ is the adjoint of $\mathcal{D}$, up to the factor $1/n$.

Lemma 4. *For each $f, g \in L^2(\mathbb{Z}_n)$ we have*

$$\frac{1}{n}\langle \mathcal{D}f, g\rangle = \langle f, \mathcal{D}^{-1}g\rangle.$$

Proof:

$$\langle f, D^{-1}g\rangle = \sum_{k=0}^{n-1} f(k)\overline{D^{-1}g(k)}$$

$$= \sum_{k=0}^{n-1} f(k)\overline{\frac{1}{n}\sum_{\ell=0}^{n-1} g(\ell)\omega^{k\ell}}$$

$$\overline{\omega} = \omega^{-1}$$

$$\omega = e^{2\pi i/n}$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} f(k)\sum_{\ell=0}^{n-1}\overline{g(\ell)}\,\omega^{-k\ell}$$

$$= \frac{1}{n}\sum_{\ell=0}^{n-1}\overline{g(\ell)}\sum_{k=0}^{n-1} f(k)\,\omega^{-k\ell}$$

$$\underbrace{\qquad\qquad}_{Df(\ell)}$$

$$= \frac{1}{n} \sum_{\ell=0}^{n-1} \overline{g(\ell)}\, Df(\ell)$$

$$= \frac{1}{n} \langle Df, g \rangle \qquad \blacksquare$$

Recall $\frac{1}{n}\langle \mathcal{D}f, g\rangle = \langle f, \mathcal{D}^{-1}g\rangle$

Parseval's identities

An immediate consequence of the adjoint lemma is Parseval's identities.

Theorem 5 (Parseval's Identities)**.** *Let $f, g \in L^2(\mathbb{Z}_n)$. Then it holds that*

(i) $\langle f, g\rangle = \frac{1}{n}\langle \mathcal{D}f, \mathcal{D}g\rangle$, *and*

(ii) $\|f\|^2 = \frac{1}{n}\|\mathcal{D}f\|^2$.

Proof of (i)

adjoint property

$$\langle f, g\rangle = \langle f, \mathcal{D}^{-1}\mathcal{D}g\rangle$$

$$= \frac{1}{n}\langle \mathcal{D}f, \mathcal{D}g\rangle.$$

Proof of (ii)

$$\|f\|^2 = \langle f, f\rangle$$

$$\overset{(i)}{=} \frac{1}{n}\langle \mathcal{D}f, \mathcal{D}f\rangle = \frac{1}{n}\|\mathcal{D}f\|^2 \quad \blacksquare$$

$$\cos\theta = \frac{x \cdot y}{\|x\| \|y\|}, \quad x, y \in \mathbb{R}$$

(i) Angle btw f and g unchanged by D up to $\frac{1}{n}$

(ii) length of f preserved by D up to $\frac{1}{n}$.

Same as rotation of coordinate system.

Parseval's identities

Remark 6. Of course, a similar statement holds for the inverse transform $\mathcal{D}^{-1}$. Indeed, Lemma 4 and Theorem 2 imply

$$\frac{1}{n}\langle f, g\rangle = \frac{1}{n}\langle \mathcal{D}\mathcal{D}^{-1} f, g\rangle = \langle \mathcal{D}^{-1} f, \mathcal{D}^{-1} g\rangle.$$

Setting $f = g$ yields $\frac{1}{n}\|f\|^2 = \|\mathcal{D}^{-1} f\|^2$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(Proof via Parseval)

Convolution

Definition 7. The *discrete cyclic convolution* of $f, g \in L^2(\mathbb{Z}_n)$, denoted $f * g$, is the function in $L^2(\mathbb{Z}_n)$ defined for each k by

$$(f * g)(k) = \sum_{j=0}^{n-1} f(j)g(k-j).$$

We note that the definition of the convolution makes use of the fact that $\mathbb{Z}_n$ is a cyclic group when $k-j$ falls outside of $0, 1, \ldots, n-1$ (i.e., the values wrap around). We leave some basic properties of the convolution to an exercise.

Exercise 8. Let $f, g, h \in L^2(\mathbb{Z}_n)$. Show that the following hold.

(i) $f * g = g * f$;

(ii) $f * (g * h) = (f * g) * h$;

(iii) $f * (g + h) = f * g + f * h$.

△

Convolution and the DFT

Lemma 9 (Convolution and the DFT)**.** *For* $f, g \in L^2(\mathbb{Z}_n)$ *we have*

$$\mathcal{D}(f * g) = \mathcal{D}f \cdot \mathcal{D}g. \tag{2}$$

Remark 10. Lemma 9 is the most important property of the DFT, that it turns convolution into multiplication. It allows us to compute convolutions with the FFT in $O(n \log n)$ operations as

$$f * g = \mathcal{D}^{-1}(\mathcal{D}f \cdot \mathcal{D}g).$$

Computing convolution the ordinary way takes $O(n^2)$ operations:

$$(f * g)(k) = \sum_{j=0}^{n-1} f(j)g(k-j).$$

The convolution property is also what allows the FFT to be used for solving PDEs numerically (all discrete derivatives are convolutions).

$$D(f*g) = Df \cdot Dg.$$

Proof: $D(f*g)(\ell) = \sum_{k=0}^{n-1} (f*g)(k)\, \omega^{-k\ell}$

$$= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} f(j)\, g(k-j)\, \omega^{-k\ell}$$

$$= \sum_{j=0}^{n-1} f(j)\, \omega^{-j\ell} \sum_{k=0}^{n-1} g(k-j)\, \omega^{j\ell - k\ell}$$

$$(*) = \sum_{j=0}^{n-1} f(j)\, \omega^{-j\ell} \underbrace{\sum_{k=0}^{n-1} g(k-j)\, \omega^{-\ell(k-j)}}_{Dg(\ell)}$$

Since $\mathbb{Z}_n$ is cyclic, we have for any $h \in L^2(\mathbb{Z}_n)$: $(**)\ \sum_{k=0}^{n-1} h(k) = \sum_{k=0}^{n-1} h(k-j), \ \forall j$

(*) $= Dg(\ell) \sum_{j=0}^{n-1} f(j)\, \omega^{-j\ell}$

$= Dg(\ell)\, Df(\ell)$ ∎

(**) $\sum_{k=0}^{n-1} h(k-j) = \underbrace{\sum_{k=j}^{n-1} h(k-j)}_{\ell = k-j} + \sum_{k=0}^{j-1} h(k-j)$

↓ $h \in \ell^2(\mathbb{Z}_n)$ is n-periodic

$= \sum_{\ell=0}^{n-j-1} h(\ell) + \underbrace{\sum_{k=0}^{j-1} h(n+k-j)}_{\ell = n+k-j}$

$k=0 \Leftrightarrow \ell = n-j$

$k = j-1 \longleftrightarrow \ell = n-1$

$$= \sum_{\ell=0}^{n-j-1} h(\ell) + \sum_{\ell=n-j}^{n-1} h(\ell)$$

$$= \sum_{\ell=0}^{n-1} h(\ell)$$

Exercise on discrete derivatives

Exercise 11. Discrete derivatives (difference quotients) can be interpreted as convolutions. Complete the following exercises.

(i) For $f \in L^2(\mathbb{Z}_n)$ define the backward difference

$$\nabla^- f(k) = f(k) - f(k-1).$$

Find $g \in L^2(\mathbb{Z}_n)$ so that $\nabla^- f = f * g$ and use this with Lemma 9 to show that $\mathcal{D}(\nabla^- f)(k) = (1 - \omega^{-k})\mathcal{D}f(k)$, where $\omega = e^{2\pi i/n}$.

(ii) For $f \in L^2(\mathbb{Z}_n)$ define the forward difference

$$\nabla^+ f(k) = f(k+1) - f(k).$$

Find $g \in L^2(\mathbb{Z}_n)$ so that $\nabla^+ f = f * g$ and use this with Lemma 9 to show that $\mathcal{D}(\nabla^+ f)(k) = (\omega^k - 1)\mathcal{D}f(k)$.

(iii) For $f \in L^2(\mathbb{Z}_n)$ define the centered difference by

$$\nabla f(k) = \frac{1}{2}(\nabla^- f(k) + \nabla^+ f(k)) = \frac{1}{2}(f(k+1) - f(k-1)).$$

Use parts (i) and (ii) to show that

$$\mathcal{D}(\nabla f)(k) = \frac{1}{2}(\omega^k - \omega^{-k})\mathcal{D}f(k) = i\sin(2\pi k/n)\mathcal{D}f(k).$$

(iv) For $f \in L^2(\mathbb{Z}_n)$, define the discrete Laplacian as

$$\Delta f(k) = \nabla^+ f(k) - \nabla^- f(k) = f(k+1) - 2f(k) + f(k-1).$$

Use parts (i) and (ii) to show that

$$\mathcal{D}(\Delta f)(k) = (\omega^k + \omega^{-k} - 2)\mathcal{D}f(k) = 2(\cos(2\pi k/n) - 1)\mathcal{D}f(k). \qquad \triangle$$