Matrix Exponentials

We have learned many different methods of solving first order differential equations, so it might be useful to have an overview of what methods we use and when we use them.

Our differential equation will be of the form

$$\dot{\mathbf{x}} = A\mathbf{x}$$

When A has non-repeated eigenvalues, either real or complex, the solution to the differential equation is

$$\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v_1} + c_2 e^{\lambda_2 t}\mathbf{v_2} + \dots$$

where λ_i are the eigenvalues and $\mathbf{v_i}$ are the corresponding eigenvectors.

It can be proven (your textbook does this) that the solution above is the same as writing $\mathbf{x} = U\Lambda_t U^{-1}\mathbf{x_0}$. (Here $\mathbf{c} = U^{-1}\mathbf{x_0}$.) where U is the eigenvector matrix and Λ_t is the diagonal matrix with entries $e^{\lambda_i t}$.

If our matrix A has repeated eigenvalues, our methods for solving are a bit different. See the treatise on repeated eigenvalues posted on the web page.

So where do matrix exponentials come in? We know that the solution to the regular differential equation $x' - ax = 0$ is $x = x_0 e^{at}$ (Try it! It's very straightforward.) So we would like our system to be analogous, and have solution

$$\mathbf{x} = e^{At}\mathbf{x_0}$$

This gives us a way to calculate e^{At}. If A has non-repeated eigenvalues, then we know the solution to the differential equation is

$$\mathbf{x} = U\Lambda_t U^{-1}\mathbf{x_0} = e^{At}\mathbf{x_0}$$

so

$$e^{At} = U\Lambda_t U^{-1}$$

This gives us a method for finding matrix exponentials in this special case.

Notice that this has no dependence on whether or not the eigenvalues are real or complex. The solution to the differential equation will always be $U\Lambda_t U^{-1}$, as long as the eigenvalues are not repeated. In the case of complex eigenvalues, the entries in the U and Λ_t matrices will have complex numbers in them.

However, sometimes it's difficult to work with complex numbers. So, to calculate the matrix exponential for a matrix with complex eigenvalues, we can use a trick, instead of working directly with the eigenvector matrix.

The trick is this: Calculate one eigenvector for the matrix. Then, let Q be a matrix with the first column being the real part of the eigenvector, and the second column being the imaginary part. For instance, if we find an eigenvector $\mathbf{v} = \binom{1+i}{1}$, then

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Then, if we multiply A on either side by Q and Q^{-1}, we get a matrix of a familiar form: $D = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$, where the eigenvalues of A are $a \pm bi$. Then, our matrix exponential will be

$$e^{At} = Qe^{Dt}Q^{-1} = Qe^{at}\begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix} Q^{-1}$$

(When we are taking an exponential of something of this form: $B = QAQ^{-1}$, $e^{Bt} = Qe^{At}Q^{-1}$. You can see this from the definition.)

Note that this is a trick. Doing this is basically equivalent to putting the solution of your differential equation into sine and cosine form, instead of keeping it in terms of complex eigenvalues and eigenvectors. You can still calculate the matrix exponential as before, but you may have some i's in your answer. If you convert these to sine and cosine form, you should get the same answer.

Also note that the only time we use the definition to calculate matrix exponentials is in a few special cases, like we did in class. In general, using the definition to calculate a matrix exponential is difficult. You shouldn't do it.

This is all well and good, but what happens when we have repeated eigenvalues? In this case, our eigenvector matrix is not invertible, in general, because we do not have linearly independent eigenvectors to put in the columns of the matrix. So in this case, we use a different trick, as in his crazy example. In this case we can't actually diagonalize our matrix, but we can get close. We are essentially multiplying by a matrix and its inverse to do this. The trick is finding the matrices to use. You should probably memorize the formulation of these matrices.