Math and Climate Seminar — IMA

MCRN

Mathematics and Climate Research Network

Joint MCRN/IMA Math and Climate Seminar

Tuesdays 11:15 – 12:05

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Budyko's Model as an Infinite Dimensional Dynamical System, II

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Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\bar{T}-T)$$

- heat capacity: R
- annual global mean surface temperature: $\frac{\partial T}{\partial t}$
- insolation: Q
- sin(latitude): y
- ice line: η
- albedo: $\alpha(y,\eta)$
- OLR: $A+BT$
- heat transport: $C(\bar{T}-T)$, where $\bar{T} = \int_0^1 T(y)dy$

$$\alpha(y,\eta) = \begin{cases} \alpha_1, & y<\eta, \\ \alpha_2, & y>\eta. \end{cases}$$

$$T = T(y,t), \quad y\in[0,1], \quad T(\cdot,t):[0,1]\to\mathbb{R}, \quad T(\cdot,t)\in X$$

What is X?

Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\bar{T}-T)$$

Fix η and look for an equilibrium solution $T_\eta^*(y)$.

$$Qs(y)(1-\alpha(y,\eta)) - \left(A+BT_\eta^*(y)\right) + C\left(\bar{T}_\eta^* - T_\eta^*(y)\right) = 0$$

Integrate

$$\int_0^1\left(Qs(y)(1-\alpha(y,\eta)) - \left(A+BT_\eta^*(y)\right) + C\left(\bar{T}_\eta^* - T_\eta^*(y)\right)\right)dy = 0$$

$$Q(1-\bar{\alpha}(\eta)) - \left(A+B\bar{T}_\eta^*\right) = 0$$

where $\bar{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)dy$

$$\bar{T}_\eta^* = \frac{Q(1-\bar{\alpha}(\eta))-A}{B}$$

Solve for GMT

Plug GMT into equilibrium condition and solve for temperature profile.

Budyko's Model

Budyko's Equilibrium

$$T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)(1-\alpha(y,\eta)) - A + C\bar{T}_\eta^*\right),$$

where $\bar{T}_\eta^* = \frac{1}{B}\left(Q(1-\bar{\alpha}(\eta))-A\right),$

$$\bar{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)dy = 0, \qquad \alpha(y,\eta) = \begin{cases} \alpha_1, & y<\eta, \\ \alpha_2, & y>\eta. \end{cases}$$

Note that, piecewise, the equilibrium is a linear function of $s(y)$.

$$T_\eta^*(y) = \begin{cases} \frac{1}{B+C}\left(Qs(y)(1-\alpha_1) - A + C\bar{T}_\eta^*\right), & y<\eta, \\ \frac{1}{B+C}\left(Qs(y)(1-\alpha_2) - A + C\bar{T}_\eta^*\right), & y>\eta. \end{cases}$$

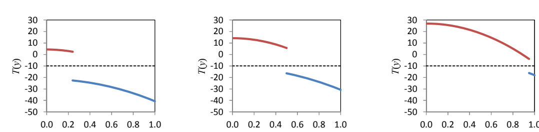
Only dependence on y.

Budyko's Model

Budyko's Equilibrium

$$T_\eta^*(y) = \begin{cases} \frac{1}{B+C}\left(Qs(y)(1-\alpha_1) - A + C\bar{T}_\eta^*\right), & y<\eta, \\ \frac{1}{B+C}\left(Qs(y)(1-\alpha_2) - A + C\bar{T}_\eta^*\right), & y>\eta. \end{cases}$$

There is an equilibrium solution for every ice boundary.



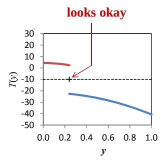
Budyko's Model

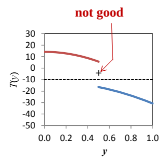
Budyko's Equilibrium

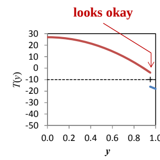
$$T_\eta^*(y) = \begin{cases} \frac{1}{B+C}\left(Qs(y)(1-\alpha_1) - A + C\overline{T}_\eta^*\right), & y < \eta, \\ \frac{1}{B+C}\left(Qs(y)(1-\alpha_2) - A + C\overline{T}_\eta^*\right), & y > \eta. \end{cases}$$

Which are the correct equilibria?
Additional condition:

$$\frac{1}{2}\left(T_\eta^*(\eta+) + T_\eta^*(\eta-)\right) = T_c = -10$$







Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C(\overline{T} - T)$$

How do we dynamically determine the "correct" solution?

But first, back to the question:
What is X, the space of functions where T lives?

Goal:
X should be small, *e.g.*, finite dimensional.
X should contain the equilibrium solutions.

Recall:
The equilibrium solutions are piecewise linear functions of the insolation distribution function $s(y)$.

What is $s(y)$?

Budyko's Model

What is $s(y)$?

$$s(y) = \frac{2}{\pi^2}\int_0^{2\pi} \sqrt{1 - \left(\sqrt{1-y^2}\sin\beta\cos\theta - y\cos\beta\right)^2}\, d\theta$$

where β = obliquity. (Current value is about 23.5°.)

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 - 0.241\left(3y^2 - 1\right) = 1 - 0.482\,p_2(y)$$

where

$$p_2(y) = \tfrac{1}{2}\left(3y^2 - 1\right)$$

(the quadratic Legendre polynomial)

Quadratic approximation for arbitrary β:

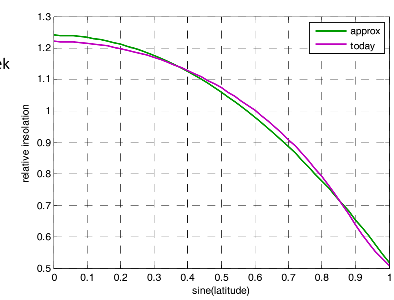
$$s(y) \approx 1 + s_2(\beta)p_2(y), \quad \text{where} \quad s_2(\beta) = \frac{5}{16}\left(-2 + 3\sin^2\beta\right)$$

Budyko's Model

What is $s(y)$?

green = quadratic approximation (Chylek & Coakley)

fuchsia = formula using obliquity of 23.5°



Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C(\overline{T} - T)$$

Back to the question:
What is X, the space of functions where T lives?

Goal:
X should be small, *e.g.*, finite dimensional.
X should contain the equilibrium solutions.

Recall:
The equilibrium solutions are piecewise linear functions of the insolation distribution function $s(y)$, which we are approximating with an even quadratic polynomial.

Proposal:

Let X be the space of functions that, piecewise, are even quadratic polynomials.

Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) + C(\overline{T} - T)$$

where $\alpha(y,\eta) = \begin{cases} \alpha_1, & y < \eta, \\ \alpha_2, & y > \eta, \end{cases}$ and $\overline{T} = \int_0^1 T(y)\,dy$

Quadratic Assumptions

$s(y) = 1 + s_2 p_2(y)$, where $p_2(y) = \frac{1}{2}(3y^2 - 1)$.

T is piecewise even and quadratic, i.e.,
X is the space of piecewise even quadratic polynomials, a four dimensional function space.

More precisely,

$$X = \mathscr{P} \times \mathscr{P} \cong \mathbb{R}^2 \times \mathbb{R}^2$$

where $\mathscr{P}$ is the space of even quadratic polynomials.

Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)\big(1-\alpha(y,\eta)\big) - (A+BT) + C(\bar{T}-T)$$

We parameterize X by introducing four new variables, w_0, z_0, w_2, and z_2 by letting

$$T(y) = \begin{cases} w_0 + \frac{1}{2}z_0 + \left(w_2 + \frac{1}{2}z_2\right)p_2(y), & y<\eta, \\ w_0 - \frac{1}{2}z_0 + \left(w_2 - \frac{1}{2}z_2\right)p_2(y), & y>\eta, \\ w_0 + w_2 p_2(\eta), & y=\eta. \end{cases}$$

$T(\eta) = \frac{1}{2}\big(T(\eta+) + T(\eta+)\big)$

CALC IV

$\alpha_0 = \frac{1}{2}(\alpha_1+\alpha_2)$

$P_2(\eta) = \frac{1}{2}(\eta^3 - \eta)$

$$\begin{aligned} R\dot{w}_0 &= Q(1-\alpha_0) - A - Bw_0 + C\left((\eta-\tfrac{1}{2})z_0 + z_2P_2(\eta)\right) \\ R\dot{z}_0 &= Q(\alpha_2-\alpha_1) - (B+C)z_0 \\ R\dot{w}_2 &= Qs_2(1-\alpha_0) - (B+C)w_2 \\ R\dot{z}_2 &= Qs_2(\alpha_2-\alpha_1) - (B+C)z_2 \end{aligned}$$

Budyko's Model

Quadratic Budyko System

$$\begin{aligned} R\dot{w}_0 &= Q(1-\alpha_0) - A - Bw_0 + C\left((\eta-\tfrac{1}{2})z_0 + z_2P_2(\eta)\right) \\ R\dot{z}_0 &= Q(\alpha_2-\alpha_1) - (B+C)z_0 \\ R\dot{w}_2 &= Qs_2(1-\alpha_0) - (B+C)w_2 \\ R\dot{z}_2 &= Qs_2(\alpha_2-\alpha_1) - (B+C)z_2 \end{aligned}$$

Note that the last three equations are independent of the first equation and of each other. Therefore, the variables z_0, w_2, and z_2 all exponentially approach their equilibrium values,

$$z_0^* = \frac{Q(\alpha_2-\alpha_1)}{B+C}, \quad w_2^* = \frac{Qs_2(1-\alpha_0)}{B+C}, \quad z_2^* = \frac{Qs_2(\alpha_2-\alpha_1)}{B+C}$$

The system reduces to a single equation,

$$R\dot{w}_0 = Q(1-\alpha_0) - A - Bw_0 + C\left((\eta-\tfrac{1}{2})z_0^* + z_2^*P_2(\eta)\right)$$

$w = w_0$

$$R\dot{w} = Q(1-\alpha_0) - A - Bw + \frac{CQ(\alpha_2-\alpha_1)}{B+C}\left(\eta - \tfrac{1}{2} + s_2P_2(\eta)\right)$$

Budyko's Model

Quadratic Budyko Equation

$$R\dot{w} = Q(1-\alpha_0) - A - Bw + \frac{CQ(\alpha_2-\alpha_1)}{B+C}\left(\eta - \tfrac{1}{2} + s_2P_2(\eta)\right)$$

or $\quad R\dot{w} = -B\big(w - F(\eta)\big)$

where

$$F(\eta) = \frac{Q(1-\alpha_0)-A}{B} + \frac{CQ(\alpha_2-\alpha_1)}{B(B+C)}\left(\eta - \tfrac{1}{2} + s_2P_2(\eta)\right)$$

As before, for each η, we have an equilibrium solution,

$$w^* = F(\eta)$$

The equilibrium temperature profile is

$$T_\eta^*(y) = \begin{cases} w^* + \frac{1}{2}z_0^* + \left(w_2^* + \frac{1}{2}z_2^*\right)p_2(y), & y<\eta, \\ w^* - \frac{1}{2}z_0^* + \left(w_2^* - \frac{1}{2}z_2^*\right)p_2(y), & y>\eta, \\ w^* + w_2^* p_2(\eta), & y=\eta. \end{cases}$$

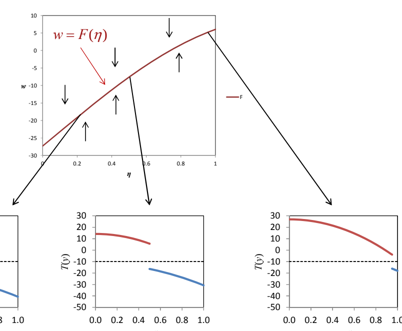
where

$$z_0^* = \frac{Q(\alpha_2-\alpha_1)}{B+C}, \quad w_2^* = \frac{Qs_2(1-\alpha_0)}{B+C}, \quad z_2^* = \frac{Qs_2(\alpha_2-\alpha_1)}{B+C}$$

Budyko's Model

Quadratic Budyko Equation

$$R\dot{w} = -B\big(w - F(\eta)\big)$$



Budyko's Model

Quadratic Budyko Equation

$$R\dot{w} = -B\big(w - F(\eta)\big)$$

Equilibrium for each η: $\quad w^* = F(\eta)$

"Correct" equilibria: choose η so that

$$T_\eta^*(\eta) = \tfrac{1}{2}\left(T_\eta^*(\eta-) + T_\eta^*(\eta+)\right) = w^* + w_2^*p_2(\eta) = T_c$$

$$w^* = -w_2^*p_2(\eta) + T_c = -\frac{Q(1-\alpha_0)}{B+C}s_2p_2(\eta) + T_c$$

$$w^* = G(\eta)$$

where

$$G(\eta) = -\frac{Q(1-\alpha_0)}{B+C}s_2p_2(\eta) + T_c$$

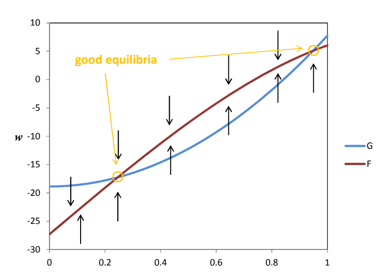
I.e., we choose η so that

$$F(\eta) = w^* = G(\eta)$$

Budyko's Model

Quadratic Budyko Equation

$$R\dot{w} = -B\big(w - F(\eta)\big)$$



Budyko's Model

Quadratic Budyko Equation

$$R\dot{w} = -B\left(w - F(\eta)\right)$$

If we don't let η change, then w simply approaches $F(\eta)$ asymptotically.

What about the dynamics of η ?

Assumption: η always changes so that

$$T(\eta) = w + w_2^* p_2(\eta) = T_c$$

(As before, we are assuming that w_2 has reached its equilibrium value.)

I.e., we constrain (η, w) to the curve

$$w = -w_2^* p_2(\eta) + T_c = G(\eta)$$

$$R\dot{w} = RG'(\eta)\dot{\eta}$$

$$R\dot{\eta} = \frac{R\dot{w}}{G'(\eta)} = \frac{-B\left(w - F(\eta)\right)}{G'(\eta)}$$

$$R\frac{d\eta}{dt} = \frac{B\left(F(\eta) - G(\eta)\right)}{G'(\eta)}$$

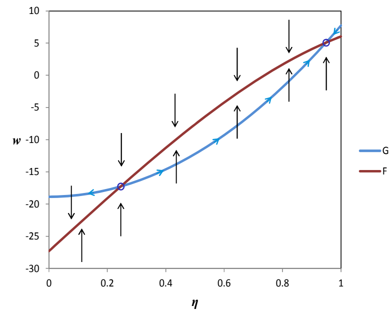
Budyko's Model

Quadratic Budyko Equation

$$R\dot{w} = -B\left(w - F(\eta)\right)$$

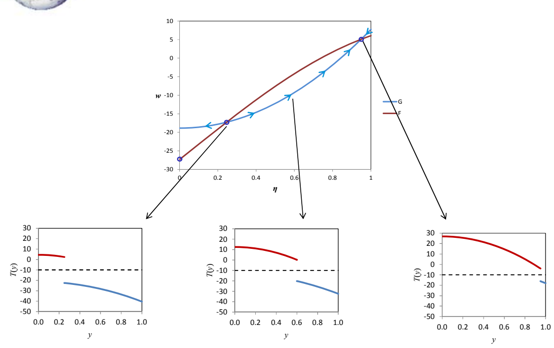
$$w = G(\eta)$$

$$R\dot{\eta} = \frac{B\left(F(\eta) - G(\eta)\right)}{G'(\eta)}$$



Budyko's Model

Quadratic Budyko Equation



Budyko's Model

Quadratic Budyko Equation

$$R\dot{w} = -B\left(w - F(\eta)\right)$$

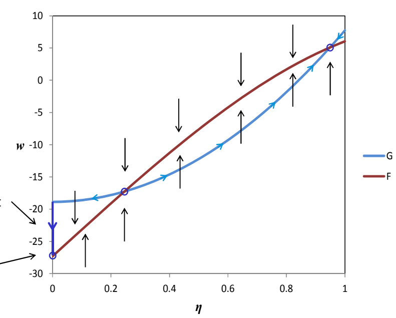
$$w = G(\eta)$$

$$R\dot{\eta} = \frac{B\left(F(\eta) - G(\eta)\right)}{G'(\eta)}$$

w continues to adjust

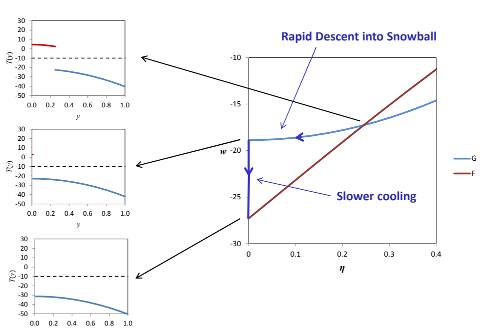
$$R\dot{w} = -B\left(w - F(\eta)\right)$$

Snowball



Budyko's Model

Quadratic Budyko Equation



Budyko's Model

Quadratic Budyko Equation

$$R\frac{\partial T}{\partial t} = \underbrace{Qs(y)}_{\text{insolation}}\underbrace{\left(1 - \alpha(y,\eta)\right)}_{\text{albedo}} - \underbrace{\left(A + BT\right)}_{\text{re-radiation}} + \underbrace{C\left(\bar{T} - T\right)}_{\text{transport}}$$

quadratic condition + invariant line

$$R\dot{w} = -B\left(w - F(\eta)\right)$$

instantaneous ice line adjustment $\left(w = G(\eta)\right)$

$$R\dot{\eta} = \frac{B\left(F(\eta) - G(\eta)\right)}{G'(\eta)}$$

outgoing longwave radiation

$$G(\eta) = -\frac{Q(1-\alpha_0)}{B+C}s_2 p_2(\eta) + T_c$$

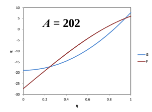
parameter dependence

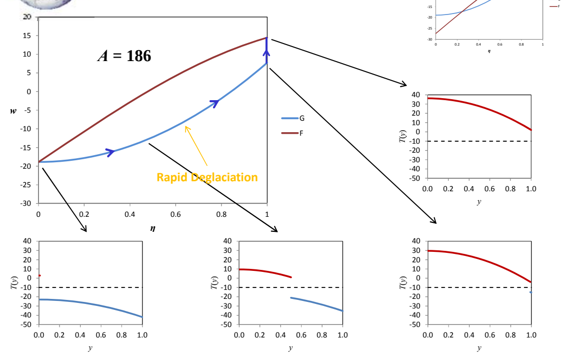
$$F(\eta) = \frac{Q(1-\alpha_0) - A}{B} + \frac{CQ(\alpha_2 - \alpha_1)}{B(B+C)}\left(\eta - \tfrac{1}{2} + s_2 P_2(\eta)\right)$$

If we increase greenhouse gasses, we decrease A, so $F(\eta)$ increases.

Budyko's Model

Quadratic Budyko Equation

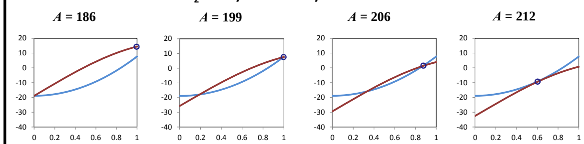




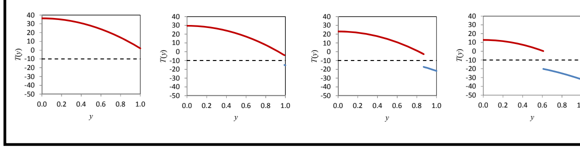
Budyko's Model

Quadratic Budyko Equation

CO_2 slowly absorbed by ocean: A increases

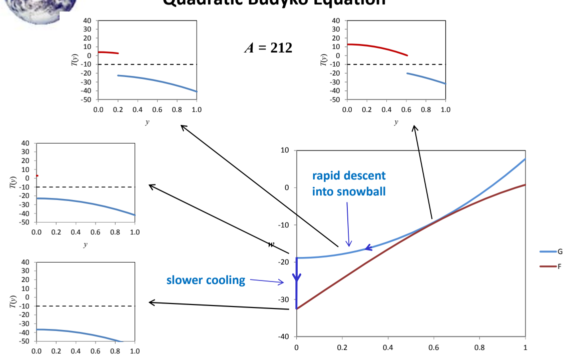


slowly following the equilibrium as A increases



Budyko's Model

Quadratic Budyko Equation



Budyko's Model

Quadratic Budyko Equation

$$R\frac{d\eta}{dt} = \frac{B\left(F(\eta)-G(\eta)\right)}{G'(\eta)}$$

$$F(\eta) = \frac{Q(1-\alpha_0)-A}{B} + \frac{CQ(\alpha_2-\alpha_1)}{B(B+C)}\left(\eta - \tfrac{1}{2} + s_2P_2(\eta)\right)$$

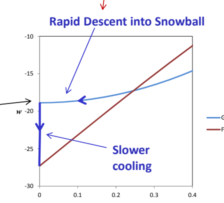
$$G(\eta) = -\frac{Q\left(1-\alpha_0\right)}{B+C}s_2p_2(\eta) + T_c$$

$$p_2(\eta) = \tfrac{1}{2}(3\eta^2 - 1)$$

$$p_2'(0) = 0, \quad G'(0) = 0$$

$$\frac{d\eta}{dt} = \infty, \quad \text{when} \quad \eta = 0$$

Problem: Too rapid



Budyko's Model

Quadratic Budyko Equation

$$R\frac{\partial T}{\partial t} = \underbrace{Qs\left(y\right)}_{\text{insolation}}\underbrace{\left(1-\alpha\left(y,\eta\right)\right)}_{\text{albedo}} - \underbrace{\left(A+BT\right)}_{\text{re-radiation}} + \underbrace{C\left(\bar{T}-T\right)}_{\text{transport}}$$

Missing: heat of fusion
(energy to melt ice)

new term

$$R\frac{\partial T}{\partial t} + R\chi\frac{d\eta}{dt} = Qs\left(y\right)\left(1-\alpha\left(y,\eta\right)\right) - \left(A+BT\right) + C\left(\bar{T}-T\right)$$

Increasing η means that energy is absorbed.
Decreasing η means that energy is released.

χ is the ratio of the energy needed to move the ice line to the energy needed to heat the surface.

Budyko's Model

Quadratic Budyko Equation

$$R\frac{d\eta}{dt} = \frac{B\left(F(\eta)-G(\eta)\right)}{G'(\eta)}$$

becomes

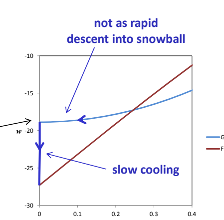
$$R\frac{d\eta}{dt} = \frac{B\left(F(\eta)-G(\eta)\right)}{G'(\eta)+\chi}$$

heat of fusion term

$$F(\eta) = \frac{Q(1-\alpha_0)-A}{B} + \frac{CQ(\alpha_2-\alpha_1)}{B(B+C)}\left(\eta - \tfrac{1}{2} + s_2P_2(\eta)\right)$$

$$G(\eta) = -\frac{Q\left(1-\alpha_0\right)}{B+C}s_2p_2(\eta) + T_c$$

$$\frac{d\eta}{dt} = \frac{B\left(F(0)-G(0)\right)}{\chi} < \infty, \quad \text{when} \quad \eta = 0$$



Budyko's Model

Quadratic Budyko Equation

$$R\frac{dw}{dt} = -B\left(w - F(\eta)\right)$$

Let's add Widiasih's equation:

$$R\frac{d\eta}{dt} = \rho\left(T(\eta) - T_c\right) = \rho\left(w - G(\eta)\right)$$

Now we don't have a constraint.
Instead we have a vector field in the plane

$$R\frac{d\eta}{dt} = \rho\left(w - G(\eta)\right)$$
$$R\frac{dw}{dt} = -B\left(w - F(\eta)\right)$$

Budyko's Model

Quadratic Budyko Equation

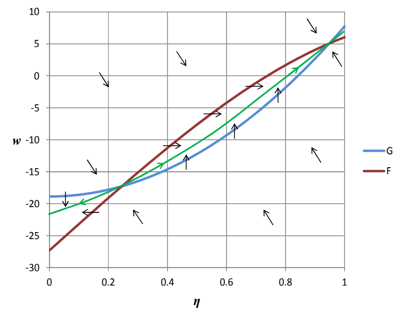
$$R\dot{\eta} = \rho\left(w - G(\eta)\right)$$
$$R\dot{w} = -B\left(w - F(\eta)\right)$$

invariant curve

$$w = \Phi(\eta)$$

$$R\dot{\eta} = \rho\left(\Phi(\eta) - G(\eta)\right)$$

$$\dot{\eta} = \varepsilon\left(\Phi(\eta) - G(\eta)\right)$$



Budyko's Model

Quadratic Budyko Equation

$$R\dot{\eta} = \rho\left(w - G(\eta)\right)$$
$$R\dot{w} = -B\left(w - F(\eta)\right)$$

Large ρ

$$w \approx G(\eta)$$

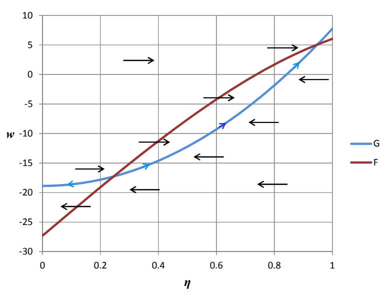
Invariant curve approaches the η isocline

$$R\dot{w} \approx -B\left(G(\eta) - F(\eta)\right)$$

$$w \approx G(\eta)$$
$$\dot{w} \approx G'(\eta)\dot{\eta}$$

$$\frac{d\eta}{dt} \approx \frac{-B\left(G(\eta) - F(\eta)\right)}{RG'(\eta)}$$

Instantaneous ice line adjustment



Budyko's Model

Quadratic Budyko Equation

$$R\dot{\eta} = \rho\left(w - G(\eta)\right)$$
$$R\dot{w} = -B\left(w - F(\eta)\right)$$

Small ρ

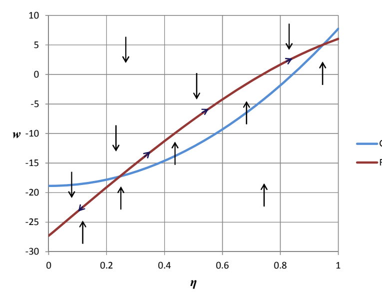
$$w \approx F(\eta)$$

Invariant curve approaches the w isocline

$$R\frac{d\eta}{dt} \approx \rho\left(F(\eta) - G(\eta)\right)$$

Reduces to Widiasih's equation:

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$



Budyko's Model

Summary

$$R\frac{\partial T}{\partial t} = \underbrace{Qs(y)}_{\text{insolation}}\left(1 - \underbrace{\alpha(y,\eta)}_{\text{albedo}}\right) - \underbrace{(A + BT)}_{\text{re-radiation}} + \underbrace{C\left(\overline{T} - T\right)}_{\text{transport}}$$

quadratic condition + invariant line

$$R\frac{dw}{dt} = -B\left(w - F(\eta)\right)$$

instantaneous ice line adjustment $(w = G(\eta))$

$$R\frac{d\eta}{dt} = \frac{B\left(F(\eta) - G(\eta)\right)}{G'(\eta)} \quad \leftarrow \rho \to \infty$$

Widiasih's system

$$R\frac{d\eta}{dt} = \rho\left(w - G(\eta)\right)$$
$$R\frac{dw}{dt} = -B\left(w - F(\eta)\right)$$

$$\rho \to 0$$

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$

Budyko's Model

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