

REIFENBERG FLATNESS AND OSCILLATION OF THE UNIT NORMAL VECTOR

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ABSTRACT. We show (under mild topological assumptions) that small oscillation of the unit normal vector implies Reifenberg flatness. We then apply this observation to the study of chord-arc domains and to a quantitative version of a two-phase free boundary problem for harmonic measure previously studied by Kenig-Toro [KT06].

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1. INTRODUCTION

The connections between the regularity of a domain (often expressed in terms of the oscillation of the unit normal) and potential theory, singular integrals and regularity for elliptic PDE has been a topic of considerable interest in mathematics (see, e.g. [Tor97]). An important object of study in this field are chord arc domains. Roughly speaking, chord arc domains exhibit interior and exterior fatness, have quantitative connectivity and have (Ahlfors) regular surface measure. If there is sufficient control on the oscillation of the unit normal (in the BMO sense) and the domain is sufficiently flat (in the sense of Reifenberg [Rei60]) we say the chord arc domain has small constant. Chord arc domains with small constant were introduced by Semmes in [Sem91a] and [Sem91b] and subsequently studied by Kenig

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and Toro (see e.g. [KT99]), Hofmann, Mitrea and Taylor (see e.g. [HMT10]) and many others. In particular, chord arc domains are a natural setting in which to consider questions of regularity for harmonic functions or harmonic measure (see e.g. [MPT14], [KT99]).

When Semmes introduced chord arc domains with small constant (see [Sem91a] and [Sem91b]) he made the assumption of *a priori* flatness and made the further restriction of working with C^2 surfaces. His focus was on operator theoretic and function theoretic properties of chord arc domains with small constant, e.g. Semmes showed the Cauchy integral operator restricted to a chord arc domain with small constant was “almost” self adjoint. In addition, Semmes expressed interest in potential theoretic characterizations. These potential theoretic characterizations were investigated by Kenig and Toro, without the *a priori* assumption of smoothness but with the *a priori* assumption of Reifenberg flatness in [KT97], [KT99] and [KT03]. As a consequence of our main theorem (Theorem 2.1), we show that the flatness hypothesis is redundant (see, e.g. Corollary 2.27), this in turn, should allow one to remove the *a priori* assumption of Reifenberg flatness from many theorems in the aforementioned works of Kenig and Toro (e.g., Theorem 4.2 in [KT99]).

Our main theorem is essentially the following: under some mild assumptions on the topology (two sided corkscrews) and surface measure (Ahlfors regularity) of a domain, small oscillation of the unit normal implies flatness in the sense of Reifenberg [Rei60]. In addition to the application mentioned above, we also use this observation to study a two-phase free boundary result first introduced by Kenig and Toro [KT06] and examined further by the first author and Hofmann [BH16]. Specifically, we can conclude a quantitative version of Theorem 1.1 in [BH16] (see our Theorem 3.4 below).

Finally, we point out that there is a robust theory of Reifenberg-type parameterizations of surfaces whose unit normal has controlled oscillation (see, e.g. [Tor95], [Mer16a] and [Mer16b]). Reifenberg-type parameterizations are a powerful tool in geometric analysis (see, e.g. [NV17]) and we believe that there should be interesting connections between our work and these parameterizations.

1.1. Definitions. Here we collect some definitions that we will need later on. The first is Ahlfors-regularity,

Definition 1.1 (Ahlfors Regular (AR)). *We say a closed set $E \subset \mathbb{R}^{n+1}$ is Ahlfors regular (AR) if there exists a constant C such that*

$$(1.2) \quad C^{-1}r^n \leq H^n(B(x, r) \cap E) \leq Cr^n$$

for all $x \in E$ and $r \in (0, \text{diam}(E))$, where H^n is the n -dimensional Hausdorff measure.

When Ω is an open set we often write $\sigma \equiv H^n|_{\partial\Omega}$, the surface measure for Ω . We may sometimes abuse terminology and say that σ is an Ahlfors-regular measure, by which we mean that (1.2) holds with $E = \partial\Omega$. When referencing a dependence on the constant C in (1.2) we will simply write AR.

Our second is corkscrew points. We need to guarantee that our domains are “fat” on both the inside and out to prevent degeneracy.

Definition 1.3 (Two-sided Corkscrew Condition). *We say an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the (M, R_0) two-sided corkscrew condition if for every $x \in \partial\Omega$ and $r \in (0, R_0)$ there exist two balls $B_1 \equiv B(x_1, r/M)$ and $B_2 \equiv B(x_2, r/M)$ such that $B_1 \subset \Omega \cap B(x, r)$ and $B_2 \subset \Omega^c \cap B(x, r)$, where Ω^c denotes the compliment of Ω . We call x_1 and x_2 interior and exterior corkscrew points respectively.*

There are several connections between Ahlfors regularity and corkscrews; if a domain, Ω , satisfies a two-sided corkscrew condition then it is automatically lower Ahlfors regular. Moreover, David and Jerison [DJ90] observed that if $\partial\Omega$ is Ahlfors regular and Ω satisfies the two-sided corkscrew condition then $\partial\Omega$ is uniformly rectifiable (see Definition 3.1 below). Our next condition is a quantitative measure of connectedness.

Definition 1.4 (Harnack Chain Condition). *Following [JK82], we say that Ω satisfies the (C, R) -Harnack Chain condition if for every $0 < \rho \leq R$, $\Lambda \geq 1$, and every pair of points $X, X' \in \Omega$ with $\delta(X), \delta(X') \geq \rho$ and $|X - X'| < \Lambda\rho$, there is a chain of open balls $B_1, \dots, B_N \subset \Omega$, $N \leq C \log_2 \Lambda + 1$, with $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ and $C^{-1} \text{diam}(B_k) \leq \text{dist}(B_k, \partial\Omega) \leq C \text{diam}(B_k)$. The chain of balls is called a ‘‘Harnack Chain’’.*

Domains which are both quantitatively fat and quantitatively connected are called NTA (Non-tangentially accessible), and were introduced by Jerison and Kenig [JK82] as a natural setting in which the boundary behavior of harmonic functions can be understood.

Definition 1.5 (NTA and Chord Arc Domains). *We say $\Omega \subset \mathbb{R}^{n+1}$ is a Non-Tangentially Accessible Domain (NTA) with constants (M, R_0) , if it satisfies the (M, R_0) -Harnack chain condition and the (M, R_0) two-sided corkscrew condition. If Ω is unbounded, we require that $\mathbb{R}^n \setminus \partial\Omega$ consists of two, non-empty, connected components. Note that if Ω is unbounded, then $R_0 = \infty$ is allowed.*

Finally, if Ω is an NTA domain whose boundary is Ahlfors regular we say Ω is a chord arc domain.

In the definition of unbounded NTA domains, it is sometimes required that $R_0 = \infty$ (see, e.g. [KT97], [KT06]). This is to obtain estimates on harmonic measure/functions at arbitrarily large scales. Since we are only interested in the geometric properties of Ω , we allow $R_0 < \infty$ even for unbounded Ω . For further discussion of unbounded domains, see Remark 2.25 below.

We need to also measure the ‘‘flatness’’ of a set. Let $E \subset \mathbb{R}^{n+1}$ be a locally compact set. For $Q \in E$ and $r > 0$ define

$$\Theta(Q, r) = \inf_L \left\{ \frac{1}{r} D(E \cap B(Q, r), L(Q, r) \cap B(Q, r)) \right\}$$

where the infimum is taken over all n -planes containing Q . Here D denotes the Hausdorff distance, that is, for $A, B \subset \mathbb{R}^{n+1}$, $D[A, B] = \sup\{d(a, B) : a \in A\} + \sup\{d(b, A) : b \in B\}$. With this in hand, we can define flatness after Reifenberg [Rei60];

Definition 1.6 (Reifenberg Flat and Vanishing Reifenberg Flat). *We say E is δ -Reifenberg flat for some $\delta > 0$ if for each compact set $K \subset \mathbb{R}^{n+1}$ there exists R_K such that*

$$\sup_{r \in (0, R_K]} \sup_{Q \in K \cap E} \Theta(Q, r) < \delta.$$

We say E is (δ, R) -Reifenberg flat if

$$\sup_{r \in (0, R]} \sup_{Q \in E} \Theta(Q, r) < \delta.$$

We say E is vanishing Reifenberg flat if for every compact set $K \subset \mathbb{R}^{n+1}$

$$\lim_{r \rightarrow 0} \sup_{Q \in E \cap K} \Theta(Q, r) = 0.$$

*Finally, we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ is δ -Reifenberg flat (or (δ, R) -Reifenberg flat, vanishing Reifenberg flat), if $\partial\Omega$ is δ -Reifenberg flat (resp. (δ, R) -Reifenberg flat, vanishing Reifenberg flat) and Ω satisfies the **separation condition**: for every compact set $K \subset \mathbb{R}^{n+1}$ there exists $R > 0$ such that for $Q \in \partial\Omega \cap K$ and $r \in (0, R]$ there exists an n -dimensional plane $L(Q, r)$ containing Q and a choice of unit normal vector to $L(Q, r)$, $\vec{n}_{Q,r}$, satisfying*

$$T^+(Q, r) = X = (x, t) = \{x + t\vec{n}_{Q,r} \in B(Q, r) : x \in L(Q, r), t > \frac{r}{4}\} \subset \Omega,$$

and

$$T^-(Q, r) = X = (x, t) = \{x + t\vec{n}_{Q,r} \in B(Q, r) : x \in L(Q, r), t < -\frac{r}{4}\} \subset \Omega^c.$$

Additionally, if Ω is unbounded we have the further requirement that $\mathbb{R}^{n+1} \setminus \partial\Omega$ consists of two connected components Ω , that $\text{int}(\Omega^c) \neq \emptyset$ and that $\partial\Omega$ is (δ_n, R) -Reifenberg flat for some $R > 0$. Here $\delta_n > 0$ is chosen small enough so that Ω is an NTA domain (up to scale $R_0 = R/10$, see Lemma 3.1 in [KT97]).

For unbounded domains, it is often usual to assume that $\partial\Omega$ is (δ_n, ∞) -Reifenberg flat. This is to ensure that Ω is an NTA domain at scale $R_0 = \infty$. Since we allow unbounded NTA domains to have local estimates, we only require that unbounded Reifenberg flat domains have local flatness. Again see Remark 2.25 below.

To simplify future proofs, let us make a quick remark on how the separation condition interacts with the two-sided corkscrew condition.

Remark 1.7. For a bounded domain Ω , we note that if $\partial\Omega$ is (δ, R_1) -Reifenberg flat (in the sense of sets) and Ω satisfies the (M, R_0) -two-sided corkscrew condition, then there exists a $\delta_0 \equiv \delta_0(M, R_0, R_1) > 0$ such that if $\delta < \delta_0$, then Ω is a (δ, R_2) -Reifenberg flat domain, where $R_2 = \min(R_0, R_1)/2$. The same holds for unbounded Ω with the additional *a priori* assumption that $\mathbb{R}^{n+1} \setminus \partial\Omega$ consists of two connected, non-empty components, one of which is Ω .

To see this, note that if δ is small enough (compared to M) and $L(Q, r)$ is the plane which best approximates $B(Q, r) \cap \partial\Omega$, then both the interior and exterior corkscrew points to Q at scale r , call them A^\pm , are not contained in the δr -slab around $L(Q, r)$, call it $S_\delta(Q, r)$. Furthermore they must be on different sides of $L(Q, r)$ (otherwise the segment between them will lie outside $S_\delta(Q, r)$ but contain a point of $\partial\Omega$, contradicting the Reifenberg flatness of $\partial\Omega$). Similarly, if there is a point, y , in $\Omega^c \cap B(Q, r)$ which is outside the $r/4$ -slab of $L(Q, r)$ on the same

side of $L(Q, r)$ as A^+ then the segment between A^+ and y must contain a point in $\partial\Omega$ outside of $S_\delta(Q, r)$, again a contradiction of Reifenberg flatness. As a similar argument holds for points in $\Omega \cap B(Q, r)$ on the same side of $L(Q, r)$ as A^- , the separation property follows.

Finally, we need a measure of control on the oscillation of a function. We introduce the classic BMO and VMO function spaces (note that if $\partial\Omega$ is Ahlfors regular, then it is a space of homogeneous type, see, e.g. [CW71], and thus much of the classical theory of these spaces extends to Ahlfors regular domains).

Definition 1.8 (BMO and VMO). *Let $\Omega \subset \mathbb{R}^{n+1}$ be a set of locally finite perimeter with $\partial\Omega$ Ahlfors regular. Let $f \in L^2_{loc}(d\sigma)$ where $\sigma = H^n|_{\partial\Omega}$, we say that $f \in BMO(d\sigma)$ if*

$$\|f\|_{BMO(d\sigma)} = \sup_{s>0} \sup_{Q \in \partial\Omega} \left(\int_{B(Q,s)} \left| f(z) - \int_{B(Q,s)} f(x) d\sigma(x) \right|^2 d\sigma(z) \right)^{\frac{1}{2}} < \infty.$$

We denote by VMO the closure of uniformly continuous functions on $\partial\Omega$ in the BMO-norm. There is also a notion of VMO_{loc} : $f \in VMO_{loc}$ if for every compact set $K \subset \mathbb{R}^{n+1}$

$$\lim_{s \rightarrow 0} \sup_{Q \in \partial\Omega \cap K} \left(\int_{B(Q,s)} \left| f(z) - \int_{B(Q,s)} f(x) d\sigma(x) \right|^2 d\sigma(z) \right)^{\frac{1}{2}} = 0.$$

Remark 1.9. We should remark that what we call VMO_{loc} (after [BH16]) is actually called VMO in some points in the literature (see, e.g. [KT99]). However, the definition given above is more suited to unbounded domains; in particular, for bounded domains the two definitions are equivalent, but for unbounded domains the ‘‘closure’’ definition controls the behavior of the function at large scales (see the discussion in [KT03]).

2. SMALL BMO NORM IMPLIES REIFENBERG FLAT

The goal of this section is to prove our main theorem, Theorem 2.1.

Theorem 2.1. *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is an open set satisfying the (M, R_0) -two-sided corkscrew condition whose boundary is Ahlfors regular. Then for every $\delta > 0$ there exists c_δ and r_δ depending only on δ , the Ahlfors regularity constant and the constants in the two-sided corkscrew condition such that if*

$$\|v\|_{BMO(d\sigma)} < c_\delta,$$

(where σ is the surface measure for Ω and v is the outer unit normal to Ω), then $\partial\Omega$ is (δ, r_δ) -Reifenberg flat. In light of Remark 1.7, we can choose c_δ small enough such that Ω also satisfies the separation condition and thus is a δ -Reifenberg flat domain.

Remark 2.2. We are grateful to Steve Hofmann for pointing out Theorem 4.19 in [HMT10], which is very closely related to our Theorem 2.1. There it is shown (as a corollary of a much larger theory) that control on the oscillation of the unit normal implies Reifenberg flatness, under the *a priori* assumption that Ω is a two-sided John domain (see [HMT10] for more details and definitions).

Our result is more general as the two-sided John condition is replaced by the larger class of two-sided corkscrew domains. We also remark that the methods of proof are completely different; in [HMT10] the John condition is used to establish a Poincaré inequality (see Proposition 4.13 there) on the boundary which in turn allows for a Semmes-type decomposition (Theorem 4.16 there). Reifenberg flatness then follows easily. In contrast, our proof is by compactness and uses only elementary real analysis estimates (along with some machinery from the theory of domains with locally finite perimeter).

Finally, we remark that in light of Theorem 2.1 and some of the subsequent corollaries, the assumption of two-sided John can be replaced by two-sided corkscrew in some of the theorems in [HMT10] (e.g. in their Theorem 4.16).

We begin the proof of Theorem 2.1 with a compactness lemma—the proof of which is essentially contained in [KT03].

Lemma 2.3 ([KT03, Theorem 4.1], [BET17, Appendix B]). *Let $M, R_0 > 0$. Suppose for each $i \in \mathbb{N}$, $\Omega^{(i)} \subset \mathbb{R}^{n+1}$ is an open set with uniformly n -Ahlfors-regular boundary, which satisfies the (M, R_0) -two sided corkscrew condition and $0 \in \partial\Omega^{(i)}$. Let $0 < r_i < \infty$ with $r_i \downarrow 0$ and set*

$$\Omega_i^+ := \frac{1}{r_i}\Omega^{(i)}, \quad \Omega_i^- := \text{int}((\Omega_i^+)^c), \quad \partial\Omega_i := \frac{1}{r_i}\partial\Omega^{(i)}.$$

Then there exists a subsequence (which may depend on the $\{r_i\}$), which we relabel such that

- (1) $\Omega_i^+ \rightarrow \Omega_\infty^+$ in the Hausdorff distance uniformly on compact sets,
- (2) $\Omega_i^- \rightarrow \Omega_\infty^- = \text{int}((\Omega_\infty^+)^c)$ in the Hausdorff distance uniformly on compact sets and
- (3) $\partial\Omega_i \rightarrow \partial\Omega_\infty^+ = \partial\Omega_\infty^-$ in the Hausdorff distance uniformly on compact sets.

Moreover, Ω_∞^+ (and Ω_∞^-) satisfies the two-sided corkscrew condition at all scales, is a set of locally finite perimeter whose topological boundary coincides with its measure theoretic boundary and $\partial\Omega_\infty^+$ is Ahlfors-regular. Finally, the corkscrew and Ahlfors-regularity constants of Ω_∞ depend only on the ambient dimension and the corkscrew and Ahlfors-regularity constants of the Ω_i .

Proof. That domains which satisfy a two-sided (M, R_0) -corkscrew condition are closed in the Hausdorff distance sense follows from arguing as in [KT03]. For a proof with full details see [BET17] Appendix B (the proof there shows that two-sided NTA domains form a closed class, but truncating the proof after Step 5/2 gives a complete argument for two-sided corkscrew domains). Elementary change of variables shows that if Ω is a (M, R_0) -corkscrew domain, then Ω/ρ is a $(M, R_0/\rho)$ -corkscrew domain. Since $r_i \downarrow 0$ it follows that for any R_1 , arbitrarily large, that Ω_i is a two-sided (M, R_1) -corkscrew domain (for large i depending on R_1, R_0). Therefore, Ω_∞^\pm satisfies the corkscrew condition at all scales with constant M .

The remaining conclusions follow exactly as in [KT03], Theorem 4.1. Briefly, that Ω is a set of locally finite perimeter follows from the compactness of BV functions (the corkscrew condition guarantees that $\chi_{\Omega_i} \rightarrow \chi_{\Omega_\infty}$ in L^1_{loc}). That the

topological boundary is the same as the measure theoretic boundary follows from the existence of corkscrews. The upper Ahlfors regularity follows from the lower-semi continuity of the surface measure and the lower Ahlfors regularity follows from the existence of corkscrew points. \square

With this lemma in hand, we can prove Theorem 2.1 by means of a compactness argument. Before continuing, we point out the primary difficulty is the lack of a ‘‘Portmanteau theorem’’ for signed measures. Recall that if μ_i are positive measures with $\mu_i \rightarrow \mu$ and $\mu(\partial A) = 0$, then $\mu_i(A) \rightarrow \mu(A)$, but this is not true for signed measures (take $\mu_i = \delta_{1-1/n} - \delta_1$ on the real line and $A = (0, 1)$ and note that the μ_i converge weakly to the zero measure).

Proof of Theorem 2.1. We proceed by contradiction. If the theorem is false, then there is a $\delta > 0$ and a sequence of domains $\Omega^{(i)}$ with uniform control on the Ahlfors regularity and corkscrew constants such that $\|\tilde{\nu}_i\|_{BMO(d\tilde{\sigma}_i)} < 1/i$, (where $\tilde{\sigma}_i$ is the surface measure for $\Omega^{(i)}$ and $\tilde{\nu}_i$ is the outer unit normal to $\Omega^{(i)}$) but $\partial\Omega_i$ fails to be $(\delta, 1/i)$ -Reifenberg flat. In particular, there are points $Q_i \in \partial\Omega^{(i)}$ and $r_i < 1/i$ such that $\Theta_{\Omega^{(i)}}(Q_i, r_i) \geq \delta$ for some $r_i < 1/i$. After a harmless translation, we assume that $Q_i \equiv 0$.

Apply Lemma 2.3 to $\Omega_i := \frac{1}{r_i}\Omega^{(i)}$. Let σ_i and ν_i be the surface measure and unit outer normal for Ω_i respectively. Similarly, let σ_∞ and ν_∞ be the surface measure and unit outer normal for Ω_∞ , the limiting domain obtained from Lemma 2.3. Recall that our assumptions imply that $\Theta_{\Omega_i}(0, 1) > \delta$, which implies, through the triangle inequality, that $\Theta_{\Omega_\infty}(0, 1/2) > \delta/2$. We will show, in fact, that Ω_∞ is a half-space, which will provide the desired contradiction.

Note that $0 \in \partial\Omega_i$ for all i and

$$\left| \int_{B(0,1)} \nu_i d\sigma_i \right| \leq \int_{B(0,1)} |\nu_i| d\sigma_i = 1.$$

Therefore, there exists a subsequence, which we relabel and fix henceforth, such that

$$\int_{B(0,1)} \nu_i d\sigma_i \rightarrow \vec{N},$$

for some vector \vec{N} , with $|\vec{N}| \leq 1$. Our goal is to show that $\nu_\infty(P) \equiv \vec{N}$ for $d\sigma_\infty$ -almost every point $P \in \partial\Omega_\infty$. Once we have done this it follows that $\Omega_\infty = \vec{N}^\perp$ and we have reached the desired contradiction (to see these details, follow the proof in [EG92] page 202).

The first step is to show that $|\vec{N}| = 1$. Using that $\|\tilde{v}_i\|_{BMO(d\tilde{\sigma}_i)} \rightarrow 0$ and the triangle inequality we have

$$\begin{aligned}
(2.4) \quad 0 &= \limsup_{i \rightarrow \infty} \int_{B(0,r_i)} \left| \tilde{v}_i(z) - \int_{B(0,r_i)} \tilde{v}_i(x) d\tilde{\sigma}_i(x) \right| d\tilde{\sigma}_i(z) \\
&= \limsup_{i \rightarrow \infty} \int_{B(0,1)} \left| v_i(z) - \int_{B(0,1)} v_i(x) d\sigma_i(x) \right| d\sigma_i(z) \\
&\geq \limsup_{i \rightarrow \infty} \int_{B(0,1)} \left| 1 - \int_{B(0,1)} v_i(x) d\sigma_i(x) \right| d\sigma_i(z) \\
&= \lim_{i \rightarrow \infty} \left| 1 - \int_{B(0,1)} v_i d\sigma_i \right| = |1 - |\vec{N}||.
\end{aligned}$$

Our next step is to prove the equivalence of limits of averages on the boundary; let $R > \rho > 0$, then by the uniform Ahlfors-regularity of the $\partial\Omega_i$,

$$\begin{aligned}
(2.5) \quad &\limsup_{i \rightarrow \infty} \left| \int_{B(0,\rho)} v_i(z) d\sigma_i(z) - \int_{B(0,R)} v_i(x) d\sigma_i(x) \right| \\
&\leq \limsup_{i \rightarrow \infty} \int_{B(0,\rho)} \left| v_i(z) - \int_{B(0,R)} v_i(x) d\sigma(x) \right| d\sigma_i(z) \\
&\leq \limsup_{i \rightarrow \infty} C \left(\frac{R}{\rho} \right)^n \int_{B(0,R)} \left| v_i(z) - \int_{B(0,R)} v_i(x) d\sigma(x) \right| d\sigma_i(z) \\
&\leq \limsup_{i \rightarrow \infty} C \left(\frac{R}{\rho} \right)^n \int_{B(0,Rr_i)} \left| \tilde{v}_i(z) - \int_{B(0,Rr_i)} \tilde{v}_i(x) d\tilde{\sigma}(x) \right| d\tilde{\sigma}_i(z) = 0
\end{aligned}$$

where we used $\|\tilde{v}_i\|_{BMO(d\tilde{\sigma}_i)} \rightarrow 0$ in the last line. Similarly, if $P \in \partial\Omega_\infty$ and $\rho > 0$, then, setting $R = 10(\rho + |P|)$, we have

$$\begin{aligned}
(2.6) \quad &\limsup_{i \rightarrow \infty} \left| \int_{B(P,\rho)} v_i(z) d\sigma_i(z) - \int_{B(0,R)} v_i(x) d\sigma_i(x) \right| \\
&\leq \limsup_{i \rightarrow \infty} \int_{B(P,\rho)} \left| v_i(z) - \int_{B(0,R)} v_i(x) d\sigma(x) \right| d\sigma_i(z) \\
&\leq \limsup_{i \rightarrow \infty} C \left(\frac{R}{\rho} \right)^n \int_{B(0,R)} \left| v_i(z) - \int_{B(0,R)} v_i(x) d\sigma(x) \right| d\sigma_i(z) \\
&\leq \limsup_{i \rightarrow \infty} C \left(\frac{R}{\rho} \right)^n \int_{B(0,Rr_i)} \left| \tilde{v}_i(z) - \int_{B(0,Rr_i)} \tilde{v}_i(x) d\tilde{\sigma}(x) \right| d\tilde{\sigma}_i(z) = 0.
\end{aligned}$$

Using (2.5), (2.6) and

$$\lim_{i \rightarrow \infty} \int_{B(0,1)} v_i d\sigma_i = \vec{N}$$

we have established that

$$(2.7) \quad \lim_{i \rightarrow \infty} \int_{B(P,R)} v_i d\sigma_i = \lim_{i \rightarrow \infty} \int_{B(0,1)} v_i d\sigma_i = \vec{N},$$

for all $P \in \partial\Omega_\infty$ and $R > 0$.

We are now ready to prove that $v_\infty(P) = \vec{N}$ for every $P \in \partial\Omega_\infty^*$ (where, $\partial\Omega_\infty^*$ is the reduced boundary of Ω_∞ , see [EG92, Chapter 5]). We collect several facts

abound sets of locally finite perimeter many of which can be found in [EG92, Chapter 5]. Recall, that if $P \in \partial^* \Omega_\infty$, then

$$\lim_{R \downarrow 0^+} \int_{B(P,R)} v_\infty d\sigma_\infty = v_\infty(P)$$

and $|v_\infty(P)| = 1$. Also $\sigma_\infty(\partial_* \Omega_\infty \setminus \partial^* \Omega_\infty) = 0$, where $\partial_* \Omega_\infty$ is the measure theoretic boundary. Since our domain has corkscrew points, $\partial_* \Omega_\infty = \partial \Omega_\infty$ and σ_∞ -almost every $P \in \partial \Omega$ is in $\partial^* \Omega_\infty$.

Let $0 < \epsilon < 1$ and $P \in \partial^* \Omega_\infty$. Choose $R > 0$ so that $\left| v_\infty(P) - \int_{B(P,R)} v_\infty(Q) d\sigma_\infty(Q) \right| < \epsilon$ and $\sigma_\infty(\partial B(P,R)) = 0$. Such an R exists by the definition of $\partial^* \Omega_\infty$ and the fact that $\sigma_\infty(\partial B(P,R)) = 0$ for all but countably many $R > 0$ (this follows from the Ahlfors regularity of σ_∞). Let $\varphi \equiv \vec{N} \xi$ where $\xi \in C_c(\mathbb{R}^{n+1})$ with $\chi_{B(P,R)} \leq \xi \leq \chi_{B(P,2R)}$ and $\|\xi - \chi_{B(P,R)}\|_{L^2(d\sigma_\infty)} < \epsilon \sigma_\infty(B(P,R))^{1/2}$ (the existence of ξ , whenever $\sigma_\infty(\partial B(P,R)) = 0$, follows from the continuity of σ_∞ at $B(P,R)$ and Urysohn's lemma).

We establish some simple bounds; in what follows C is a constant that only depends on the (uniform) Ahlfors-regularity bounds on Ω_i and Ω_∞ . First, we see that by the L^2 bound for $\xi - \chi_{B(P,R)}$ we have

$$(2.8) \quad \left| \int \varphi \cdot v_\infty d\sigma_\infty - \vec{N} \cdot \int_{B(P,R)} v_\infty d\sigma_\infty \right| \leq \sigma_\infty(B(P,2R))^{1/2} \|\xi - \chi_{B(P,R)}\|_{L^2(d\sigma_\infty)} < C \epsilon \sigma_\infty(B(P,R))$$

where we used the Ahlfors-regularity of σ_∞ in the last inequality.

Also, since $\left| v_\infty(P) - \int_{B(P,R)} v_\infty(Q) d\sigma_\infty(Q) \right| < \epsilon$ we may conclude from (2.8)

$$(2.9) \quad \left| \int \varphi \cdot v_i d\sigma_i - \sigma_\infty(B(P,R)) \vec{N} \cdot v_\infty \right| < (C+1) \epsilon \sigma_\infty(B(P,R)) \leq C \epsilon \sigma_\infty(B(P,R)).$$

Next, by Lemma 2.3 (in particular the fact that $\chi_{\Omega_i} \rightarrow \chi_{\Omega_\infty}$ in L^1_{loc}), we have that $v_i d\sigma_i \rightarrow v_\infty d\sigma_\infty$; to see this we need only note that $[d\chi_{\Omega_i}] = v_i d\sigma_i$ and $[d\chi_{\Omega_\infty}] = v_\infty d\sigma_\infty$ since the measure theoretic boundaries of the sets Ω_i and Ω_∞ coincide with the topological boundaries of Ω_i and Ω_∞ respectively (see [EG92, Chapter 5]). Therefore, there exists an i_0 (which depends on φ) such that for all $i \geq i_0$

$$(2.10) \quad \left| \int \varphi \cdot v_i d\sigma_i - \int \varphi \cdot v_\infty d\sigma_\infty \right| < \epsilon \sigma_\infty(B(P,R)).$$

We claim that, perhaps after adjusting i_0 , we have for all $i \geq i_0$,

$$(2.11) \quad \int \varphi \cdot v_i d\sigma_i + \epsilon \sigma_i(B(P,R)) > \vec{N} \cdot \int_{B(P,R)} v_i d\sigma_i,$$

or, equivalently,

$$(2.12) \quad \int_{B(P,2R) \setminus B(P,R)} \xi \vec{N} \cdot v_i d\sigma_i + \epsilon \sigma_i(B(P,R)) > 0,$$

where we used that $\chi_{B(P,R)} \leq \xi \leq \chi_{B(P,2R)}$. Indeed, we have a stronger statement, there exists i_0 such that for all $i \geq i_0$,

$$(2.13) \quad \sigma_i(\{z \in B(P, 2R) : \nu_i(z) \cdot \vec{N} < 0\}) < \frac{\epsilon}{2} \sigma_i(B(P, R)).$$

To prove (2.13) we have that $|\nu_i| < 1$ σ_i a.e. so that $1 - \vec{N} \cdot \nu_i \geq 0$, σ_i -a.e., and by (2.7) if i is sufficiently large

$$(2.14) \quad \left| \int_{B(P, 2R)} \vec{N} \cdot \nu_i d\sigma_i - 1 \right| < c\epsilon,$$

where c is a small constant depending on the uniform Ahlfors regularity to be chosen momentarily. Since $1 - \vec{N} \cdot \nu_i \geq 0$, σ_i -a.e., (2.14) implies

$$\left| \int_{B(P, 2R)} 1 - \vec{N} \cdot \nu_i d\sigma_i \right| < c\epsilon,$$

so that by Chebyshev's inequality

$$\sigma_i(\{z \in B(P, 2R) : 1 - \vec{N} \cdot \nu_i(z) > 1\}) \leq \sigma_i(B(P, 2R))c\epsilon < \frac{\epsilon}{2} \sigma_i(B(P, R)),$$

by choice of c . This clearly implies (2.13). Now we make the observation that

$$\begin{aligned} \limsup_{i \rightarrow \infty} \sigma_i(B(P, R)) &= \limsup_{i \rightarrow \infty} \sigma_i(B(P, R)) \vec{N} \cdot \int_{B(P, R)} \nu_i d\sigma_i \\ &= \limsup_{i \rightarrow \infty} \vec{N} \cdot \int_{B(P, R)} \nu_i d\sigma_i. \end{aligned}$$

Thus, combining (2.11), (2.10) and (2.8) we obtain

$$(2.15) \quad \begin{aligned} (1 - \epsilon) \limsup_{i \rightarrow \infty} \sigma_i(B(P, R)) &< \limsup_{i \rightarrow \infty} \int \varphi \cdot \nu_i d\sigma_i \\ &\leq \int \varphi \cdot \nu_\infty d\sigma_\infty + \epsilon \sigma_\infty(B(P, R)) \\ &\leq \sigma_\infty(B(P, R))(\vec{N} \cdot \nu_\infty(P) + C\epsilon). \end{aligned}$$

By the lower semicontinuity of the total variation of BV functions we have

$$\sigma_\infty(B(P, R)) \leq \liminf_{i \rightarrow \infty} \sigma_i(B(P, R)) \leq \limsup_{i \rightarrow \infty} \sigma_i(B(P, R))$$

so that (2.15) implies

$$\frac{\vec{N} \cdot \nu_\infty(P) + C\epsilon}{1 - \epsilon} > 1$$

for any $\epsilon \in (0, 1)$. It follows that $\nu_\infty(P) = \vec{N}$. \square

Remark 2.16. It is interesting to note (particularly in the setting of Corollary 2.22) that using the same analysis above one can show $\sigma_i \rightarrow \sigma_\infty$. Recall that we had already shown $\nu_\infty \equiv \vec{N}$, taking $f \in C_c(\mathbb{R}^n)$ with $\text{supp } f \subset B(0, R)$ we show

$$(2.17) \quad \lim_{i \rightarrow \infty} \left| \int (f \nu_i) \cdot \nu_i d\sigma_i - \int (f \vec{N}) \cdot \vec{N} d\sigma_\infty \right| = 0$$

Since f is arbitrary and $|v_i| = 1$, σ_i -a.e., this will show $\sigma_i \rightarrow \sigma_\infty$. Using the triangle inequality we have that the left hand side of (2.17) is bounded by

$$\begin{aligned}
(2.18) \quad & \limsup_{i \rightarrow \infty} \left| \int (fv_i) \cdot v_i d\sigma_i - \int (f\vec{N}) \cdot v_i d\sigma_i \right| \\
& + \limsup_{i \rightarrow \infty} \left| \int (f\vec{N}) \cdot v_i d\sigma_i - \int (f\vec{N}) \cdot \vec{N} d\sigma_\infty \right| \\
& = \limsup_{i \rightarrow \infty} \left| \int (fv_i) \cdot v_i d\sigma_i - \int (f\vec{N}) \cdot v_i d\sigma_i \right| \\
& \leq \limsup_{i \rightarrow \infty} \sigma_i(B(0, R)) \|f\|_\infty \int_{B(0, R)} (1 - \vec{N} \cdot v_i) d\sigma_i = 0
\end{aligned}$$

where the equality comes from the weak convergence $v_i d\sigma_i \rightarrow v_\infty d\sigma_\infty$ and in the last line we used (2.7), the uniform Ahlfors regularity property and that $1 - \vec{N} \cdot v_i \geq 0$, σ_i -a.e.

Remark 2.19. We remark that in Theorem 2.1 the constant c_δ does not depend on R_0 in the two sided corkscrew condition while r_δ does. For this reason we may often reduce (by scaling) to the case $R_0 = 1$. We also notice that control on the oscillation of v_i centered around $Q_i \in \partial\Omega_i$ gives control on the flatness of $\partial\Omega_i$ around Q_i . We illustrate these observations with the following two refinements/Corollaries of Theorem 2.1.

Corollary 2.20 (A Pointwise Version of Theorem 2.1). *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is an open set satisfying the (M, R_0) -two-sided corkscrew condition whose boundary is Ahlfors regular. Define for $Q \in \partial\Omega$ and $R > 0$*

$$(2.21) \quad \|v\|_*(Q, R) := \sup_{0 < s < R} \left(\int_{B(Q, s)} \left| v(z) - \int_{B(Q, s)} v(y) d\sigma(y) \right|^2 d\sigma(z) \right)^{\frac{1}{2}}.$$

Then for every $\delta > 0$ there exists c_δ and r_δ depending only on δ , the Ahlfors regularity constant and the constants in the two-sided corkscrew condition such that if

$$\|v\|_*(Q, 1) < c_\delta,$$

then

$$\sup_{r \in (0, r_\delta)} \Theta(Q, r) < \delta.$$

Proof. The proof is exactly the same as Theorem 2.1. Proceeding in a way as to obtain a contradiction, we suppose there exist domains $\Omega^{(i)}$ satisfying the hypotheses above and $Q_i \in \Omega^{(i)}$ with $\|\tilde{v}_i\|_*(Q_i, 1) < 1/i$ but $\Theta(Q_i, r_i) \geq \delta$ for some $r_i < 1/i$ (here \tilde{v}_i is the unit outer normal to $\Omega^{(i)}$ and the norm $\|\cdot\|_*$ is taken with respect to $\tilde{\sigma}_i$ to surface measure to $\Omega^{(i)}$). Again, without loss of generality we may assume $Q_i \equiv 0$. We then proceed exactly as in Theorem 2.1 noting that the analysis hinges on the following estimate for all $R > 0$,

$$\limsup_{i \rightarrow \infty} \int_{B(0, Rr_i)} \left| \tilde{v}_i(z) - \int_{B(0, Rr_i)} \tilde{v}_i(x) d\tilde{\sigma}(x) \right| d\tilde{\sigma}_i(z) = 0.$$

The expression indexed by i in the limit superior is less than $1/i$ provided that $Rr_i < 1$, which always occurs for i large enough.

□

Finally, we have that the unit normal in VMO implies vanishing Reifenberg flatness.

Corollary 2.22. *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a open set satisfying the (M, R_0) -two-sided corkscrew condition whose boundary is Ahlfors regular. If $v \in \text{VMO}_{loc}(d\sigma)$ where σ is the surface measure for Ω and v is the outer unit normal to Ω , then for every compact set $K \subset \mathbb{R}^{n+1}$*

$$\lim_{r \rightarrow 0} \sup_{Q \in \partial\Omega \cap K} \Theta(Q, r) = 0,$$

where $\Theta(Q, r)$ is associated to the set $\partial\Omega$.

Proof. Fix a compact set K and $\delta > 0$. Using the definition of VMO_{loc} we let $s > 0$ be such that

$$\sup_{Q \in K} \|v\|_*(Q, s) < c_\delta,$$

where c_δ is as in Corollary 2.20. Fix $Q \in K$. Set $\tilde{\Omega} := \frac{\Omega - Q}{s}$ and let \tilde{v} and $\tilde{\sigma}$ be the unit normal and surface measure to $\tilde{\Omega}$ respectively. It follows from a change of variables that $\|\tilde{v}\|_*(Q, 1) < c_\delta$ and hence by Corollary 2.20 we have

$$\sup_{r \in (0, r_\delta)} \theta_{\tilde{\Omega}}(0, r) < \delta,$$

where $\theta_{\tilde{\Omega}}$ is associated to $\partial\tilde{\Omega}$. Dilating and translating we have that

$$\sup_{r \in (0, sr_\delta)} \theta(Q, r) < \delta.$$

Since sr_δ is uniform over $Q \in K$ we have shown $\limsup_{r \rightarrow 0} \sup_{Q \in \partial\Omega \cap K} \Theta(Q, r) < \delta$. As $\delta > 0$ is arbitrarily small, the corollary follows. □

2.1. Applications to Chord-Arc Domains with Small Constant. Let us recall the definition of a Chord Arc Domain with Small Constant (or δ -Chord Arc Domain) introduced by Kenig and Toro (see, e.g. Definitions 1.10 and 1.11 in [KT03]).

Definition 2.23 (δ -Chord Arc Domain and Vanishing Chord Arc Domain). *We say a domain, $\Omega \subset \mathbb{R}^{n+1}$, is a δ -chord arc domain (or chord arc domain with small constant) if Ω is a δ -Reifenberg flat chord arc domain and for each compact set $K \subset \mathbb{R}^{n+1}$ there exists a $R > 0$ such that*

$$\sup_{Q \in \partial\Omega \cap K} \|v\|_*(Q, R) < \delta,$$

where v is the unit outer normal to the boundary and the norm $\|\cdot\|_*$ (recall (2.21)) is with respect to σ , the surface measure of $\partial\Omega$.

We say a domain Ω is a chord arc domain with vanishing constant if it is a chord arc domain with small constant and for each compact set $K \subset \mathbb{R}^{n+1}$

$$(2.24) \quad \lim_{r \rightarrow 0} \sup_{Q \in \partial\Omega \cap K} \|v\|_*(Q, r) = 0.$$

That is to say, $v \in \text{VMO}_{loc}(d\sigma)$.

A consequence of our Theorem 2.1 and Corollary 2.22 is that the *a priori* assumption of Reifenberg flatness (assuming the presence of corkscrews) in Definition 2.23 is redundant. We should remark that “quantitatively” this is not precisely true; Theorem 2.1 does not guarantee that a chord arc domain with $\|v\|_{\text{BMO}} < \delta$ is a δ -chord arc domain, only that it is a ε -chord arc domain for some $\varepsilon = \varepsilon(\delta) > 0$. However, for most applications the precise value of $\delta > 0$ is not particularly important (see, e.g. Corollary 5.2 in [KT97]).

However, before we can continue we make a quick remark as to the complications which arise when we consider unbounded chord arc domains.

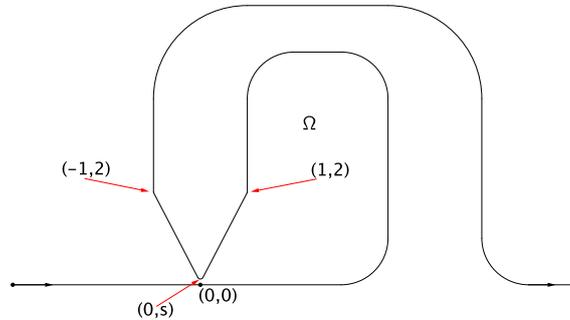
Remark 2.25. When working with unbounded domains, there is an issue that δ -Reifenberg flatness of $\partial\Omega$ and $v \in \text{VMO}_{loc}$ are conditions which hold at scales that are merely uniform over compacta, whereas the NTA conditions are required to hold at uniform scales throughout the domain (see Example 2.26 below for a potentially problematic domain). Therefore, in order to ensure the unbounded domains in the results below are NTA, we must assume, *a priori*, some global flatness (or smallness of BMO norm). We note that the need for global control is also why (δ, R) -Reifenberg flatness is included in the definition of unbounded Reifenberg flat domains.

The following example illustrates the issues discussed in Remark 2.25. The example is a modification of an example provided to the first author by Steve Hofmann while working on [BH16].

Example 2.26. For $n \geq 2$, there exists a domain $\Omega \subset \mathbb{R}^n$ with the following properties:

- (1) $\partial\Omega$ is AR,
- (2) $\Omega_{ext} = \text{int}(\Omega^c)$ is non-empty and consists of one connected component,
- (3) Ω satisfies the two-sided corkscrew condition,
- (4) the outer normal to Ω , ν , satisfies $\nu \in \text{VMO}_{loc}$,
- (5) $\partial\Omega$ is vanishing Reifenberg flat and
- (6) Ω fails the (C, R) -Harnack chain condition for any $C, R > 0$.

We describe the domain now and leave it to the reader to verify properties (1) - (6). Given $s \in (0, 1)$ we let $\Gamma_s \subset \mathbb{R}^2$ be the curve pictured below.



Here the only portion of Γ_s that depends on s is the the curve, C_s , connecting $(-1, 2)$ to $(1, 2)$, the rest of the curve is made up of (horizontal and vertical) line segments and quarter-circle arcs. We give a brief description of C_s so the reader may verify properties (1) - (5). C_s is a “smoothed” version of the curve given by the line segment from $(1, 2)$ to $(0, s)$ followed by the line segment from $(0, s)$ to $(-1, 2)$. We construct C_s from circular arcs from $\partial B((1 - \frac{s}{2}, 2), \frac{s}{2})$, $\partial B((0, \frac{3s}{2}), \frac{s}{2})$ and $\partial B((-1 + \frac{s}{2}, 2), \frac{s}{2})$, and two line segments, one segment parallel to the the line through $(1 - \frac{s}{2}, 2)$ and $(0, \frac{3s}{2})$, and the other parallel to the line through $(-1 + \frac{s}{2}, 2)$ and $(0, \frac{3s}{2})$ in such a way that Γ_s is C^1 . To ensure corkscrew points for Ω , is it important to see that C_s stays above $y = |x|$.

Now, we construct Ω . We use the convention that $C \cup C'$ means C followed by C' , where we attach the beginning point of C' to the endpoint of C and allow our curves to be defined up to translation. Let Γ be the curve of infinite length given by

$$\Gamma := \dots \cup \Gamma_{1/4} \cup \Gamma_{1/3} \cup \Gamma_{1/2} \cup \Gamma_{1/3} \cup \Gamma_{1/4} \cup \dots$$

Clearly, $\mathbb{R}^2 \setminus \Gamma$ consists of two non-empty connected components and we let Ω be as pictured above. Conditions (1) and (2) are satisfied readily. Some elementary but tedious calculations show that (3) holds. To see that (4) and (5) hold, note that at every point, up to rotation, Ω is locally given by the region above the graph of a C^1 function and in the definition of vanishing Reifenberg flat and VMO_{loc} , we are checking that a condition is satisfied on all compact sets (but not uniformly). The failure of the (C, R) -Harnack chain condition follows from the pinching that occurs in Γ_s near $(0, 0)$ as s tends to zero (and C_s stays above $y = |x|$). We can use this example in \mathbb{R}^2 to obtain a similar example in \mathbb{R}^n for $n \geq 3$ by taking $\Omega_2 \times \mathbb{R}^{n-2}$, where Ω_2 is the example constructed here (in \mathbb{R}^2).

Corollary 2.27. *Let $\delta \in (0, \delta_n]$. Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a domain satisfying the (M, R_0) -two-sided corkscrew condition whose boundary is Ahlfors regular. If Ω is unbounded additionally assume that $\mathbb{R}^{n+1} \setminus \partial\Omega$ consists of two nonempty connected components. There exists $c_\delta < \delta$ depending only on M, δ and the AR constant such that if $\sup_{Q \in \partial\Omega} \|\tilde{\nu}\|_*(Q, R_1) < c_\delta$ for some R_1 , then Ω is a δ -chord arc domain with constants depending on M, δ, R_0, R_1 and the AR constant.*

Proof. Let c_δ be as in Corollary 2.20 (we may assume $c_\delta < \delta$) and $Q \in \partial\Omega$ be arbitrary. Set $R := \min\{R_0, R_1, 1\}$. Define $\tilde{\Omega} := \frac{\Omega - Q}{R}$, then $\tilde{\Omega}$ satisfies the $(M, 1)$ corkscrew condition and $\partial\tilde{\Omega}$ is AR with the same AR constant as Ω . Moreover, $\tilde{\nu}$, the unit outer normal to $\tilde{\Omega}$ satisfies $\|\tilde{\nu}\|_*(0, 1) < c_\delta$. Applying Corollary 2.20, we obtain the existence of r_δ such that $\sup_{r \in (0, r_\delta)} \Theta_{\tilde{\Omega}}(0, r) < \delta < \delta_n$. Scaling and translating back to Ω and noting that Q was arbitrary we obtain

$$\sup_{r \in (0, Rr_\delta)} \sup_{Q \in \partial\Omega} \Theta_{\tilde{\Omega}}(Q, r) < \delta < \delta_n.$$

Thus, Ω is (δ, Rr_δ) -Reifenberg flat. \square

Corollary 2.22 implies a similar result for vanishing chord arc domains. Note that for unbounded domains this does not improve Definition 2.23 substantively.

Corollary 2.28. *Let Ω be a domain, whose boundary is Ahlfors regular and which satisfies the (M, R_0) -two-sided corkscrew condition. If Ω is unbounded, we assume that $\mathbb{R}^{n+1} \setminus \partial\Omega$ consists of two nonempty connected components. Let c_{δ_n} be as in Corollary 2.20. Then any of the following imply that Ω is a Vanishing chord arc domain.*

- (1) $v \in VMO_{loc}$ and $\partial\Omega$ is bounded.
- (2) $\|v\|_{BMO} < c_{\delta_n}$ and $v \in VMO_{loc}$.
- (3) $\sup_{Q \in \partial\Omega} \|v\|_*(Q, R_1) < c_{\delta_n}$ for some $R_1 > 0$ and $v \in VMO_{loc}$.

Proof. By definition (1) \implies (3) and (2) \implies (3). So, we may assume that (3) holds. Corollary 2.27 then shows that Ω is a δ_n -chord arc domain and clearly for any compact set K we have

$$\lim_{r \rightarrow 0} \sup_{Q \in \partial\Omega \cap K} \|v\|_*(Q, r) = 0.$$

The result follows by Corollary 2.22. \square

3. APPLICATION TO A TWO-PHASE PROBLEM FOR HARMONIC MEASURE

In the sequel, we will assume $n \geq 2$. In this section, we apply the results of the previous section to a two-phase free boundary problem for harmonic measure, originally studied by Kenig-Toro in [KT06]. Our approach however, will be a quantified version of that of the first author with Hofmann [BH16]. In particular, as in [BH16] we avoid any *a priori* assumption on topology.

Let us introduce the necessary definitions and notation so that we may state the free boundary problem. We will assume that Ω^\pm are uniformly rectifiable (UR) domains (first introduced by Hofmann, Mitrea and Taylor in [HMT10]). To properly define UR domains, we must first recall what it means for a set to be uniformly rectifiable. We note that the following “definition” is actually two (quite deep) theorems and that a proper introduction to uniformly rectifiable sets would first give a more geometric characterization (see, e.g. [DS91, DS93]). However, for our purposes, the following characterization of UR sets is most useful:

Definition 3.1 (UR). (*aka uniformly rectifiable*). *Let $E \subset \mathbb{R}^{n+1}$ be an n -dimensional Ahlfors regular (hence closed) set with surface measure σ . Then E is uniformly rectifiable (UR) if and only if the Riesz transform operator, \mathcal{R} is L^2 bounded with respect to surface measure, in the sense that*

$$(3.2) \quad \sup_{\varepsilon > 0} \|\mathcal{R}_\varepsilon f\|_{L^2(E, \sigma)} \leq C \|f\|_{L^2(E, \sigma)},$$

(see Definition 3.11 for a definition of \mathcal{R} and \mathcal{R}_ε). That uniform rectifiability (defined in a geometric sense) implies the Riesz transforms are bounded is due to David and Semmes, [DS91]. The converse is due to [MMV96] when $n = 1$, and [NTV14] in general.

The constant $C > 0$ in (3.2) and the constant implicit in the Ahlfors regularity determine the “UR character” of E . Below, we will denote the dependence of a constant, K , on the UR character by $K(UR)$.

We can now define a UR domain:

Definition 3.3 (UR domain, see [HMT10]). *We will say that a domain Ω is a UR domain if $\partial\Omega$ is UR, and if the measure theoretic boundary $\partial_*\Omega$ (see [EG92, Chapter 5]) satisfies $\sigma(\partial\Omega \setminus \partial_*\Omega) = 0$.*

Note, in particular, that if an Ahlfors regular domain satisfies the two-sided corkscrew conditions then it is a UR domain (that a domain which satisfies the two-sided corkscrew condition with Ahlfors regular boundary has a UR boundary is a result of David and Jerison [DJ90]). Additionally, one should note that if $\Omega \subset \mathbb{R}^{n+1}$ is a set of locally finite perimeter then the measure theoretic boundary and the reduced boundary differ by a set of H^n measure zero, so it then follows that the measure theoretic boundary has full measure if and only if the reduced boundary has full measure (see [EG92, Section 5.8]).

For any domain Ω with ADR boundary and surface measure $\sigma = \mathcal{H}^n|_{\partial\Omega}$, we adopt the notation for $r > 0$

$$\|f\|_*(r) := \sup_{Q \in \partial\Omega} \|f\|_*(Q, r),$$

where $\|f\|_*(Q, r)$ is as in (2.21).

We can now state our theorem (we have two theorems, one for finite and the other for infinite pole):

Theorem 3.4 (Quantified version of [BH16, Theorem 1.1]). *Let $\Omega^+ \subset \mathbb{R}^{n+1}$ and $\Omega^- = \mathbb{R}^{n+1} \setminus \overline{\Omega^+}$ be connected UR domains with common boundary $\partial\Omega^+ = \partial\Omega^-$ and $\text{diam}(\partial\Omega^+) < \infty$. Let $X^+ \in \Omega^+$ and $X^- \in \Omega^-$ be such that $k^+ = \frac{d\omega^{X^+}}{d\sigma}$ and $k^- = \frac{d\omega^{X^-}}{d\sigma}$ exist. Let $\delta > 0$ and $r_0 \in (0, \text{diam}(\partial\Omega^+))$. There exists*

$$\eta = \eta(\delta, n, UR, r_0, \delta(X^+), \delta(X^-), \text{diam}(\partial\Omega^+)) > 0$$

such that if

$$\|\log k^+\|_*(r_0), \|\log k^-\|_*(r_0) < \eta,$$

then Ω^+ and Ω^- are (δ) -chord arc domains.

We state the next theorem without proof (see Remark 3.6). Note we assume that the domains are NTA domains from the outset, as it is not clear how to even define the Poisson kernel with pole at infinity without this assumption.

Theorem 3.5 (Quantified version of [BH16, Theorem 1.2 (2)]). *Let $\Omega^+ \subset \mathbb{R}^{n+1}$ and $\Omega^- = \mathbb{R}^{n+1} \setminus \overline{\Omega^+}$ be (M, ∞) -chord arc domains with common boundary $\partial\Omega^+ = \partial\Omega^-$ and $\text{diam}(\partial\Omega^+) = \infty$. Then for any $\delta > 0$ there exists $\eta = \eta(n, AR, M, \delta) > 0$ such that if*

$$\|\log k^+\|_{BMO(d\sigma)}, \|\log k^-\|_{BMO(d\sigma)} < \eta$$

then Ω^+ and Ω^- are δ -chord arc domains. Here k^+ and k^- are the Poisson kernels with pole at infinity for the domains Ω^+ and Ω^- respectively.

A few remarks are in order.

Remark 3.6. 1) We omit the proof of Theorem 3.5 because it follows from [BH16, Theorem 1.2 (2)] and Corollary 2.20 in much the same way as Theorem 3.4. In fact, the hypothesis that Ω^\pm are (M, ∞) -chord arc domains allows one to immediately

apply the ‘‘CFMS’’ estimates [CFMS81], which makes the proof of Theorem 3.5 simpler than that of Theorem 3.4 (see [BH16] for more details).

2) To simplify matters, we will prove Theorem 3.4 when $\text{diam}(\partial\Omega^+) = 1$. For this reason, we will state many of the lemmas below for $\text{diam}(\partial\Omega^+) = 1$. A simple scaling argument recovers the general case.

3) We note that [KT97] refers to (δ, R) -chord arc domains which are simply δ -chord arc domains where the flatness and the oscillation of the unit normal are controlled up to scale $R > 0$ globally. In Theorem 3.4 above (and in Definition 2.23) we follow the lead of [KT03] and suppress the scale. However, our methods allow us to keep track of the scale at which the flatness and oscillation are controlled and we try to make that clear in the proof below.

To prove Theorem 3.4, we hope to employ Corollary 2.20. Therefore, we must produce the two-sided corkscrew condition. The following lemma, while a simple consequence of observations in [AMTV16] and [AHM⁺17], may be of independent interest.

Lemma 3.7. *Let $\Omega^+ \subset \mathbb{R}^{n+1}$ and $\Omega^- = \mathbb{R}^{n+1} \setminus \overline{\Omega^+}$ be connected UR domains in \mathbb{R}^{n+1} with common boundary $\partial\Omega^+ = \partial\Omega^-$. Let $C > 0$ and suppose $X^\pm \in \Omega^\pm$ are such that*

$$(3.8) \quad \omega^{X^\pm}(\Delta(Q, r)) \leq C\omega^{X^\pm}(\Delta(Q, r/2))$$

for all $Q \in \partial\Omega^+$ and $0 < r < \frac{1}{4} \min\{\delta(X^+), \delta(X^-), \text{diam}(\partial\Omega^+)\}$. That is, ω^{X^\pm} satisfy a doubling condition.

Then there exists $M, R_0 > 0$, depending on $n, UR, \delta(X^+), \delta(X^-)$ and the constant C in (3.8) such that Ω^\pm satisfy the two-sided (M, R_0) -corkscrew condition.

Proof. Set $\partial\Omega := \partial\Omega^+$. First, we claim that for all $\epsilon > 0$, $Q \in \partial\Omega$ and $r \in (0, \text{diam}(\partial\Omega^+))$ there exists $\tilde{Q} \in \partial\Omega$ and $\tilde{r} \approx_\epsilon r$ with $B(\tilde{Q}, \tilde{r}) \subset B(Q, r)$ and an affine n -plane $V(\tilde{Q}, \tilde{r})$ satisfying

$$(3.9) \quad D[V(\tilde{Q}, \tilde{r}) \cap B(\tilde{Q}, \tilde{r}); \partial\Omega \cap B(\tilde{Q}, \tilde{r})] < \epsilon\tilde{r}.$$

Indeed, the bilateral weak geometric lemma (BWGL) [DS93, Theorem 2.4] guarantees the existence of \tilde{Q}, \tilde{r} and $V(\tilde{Q}, \tilde{r})$. The BWGL states, if $\partial\Omega$ is UR, the failure of (3.9) is quantified by a Carleson packing condition. Then a pigeon-hole argument yields \tilde{Q}, \tilde{r} and $V(\tilde{Q}, \tilde{r})$, see the discussion following [AHM⁺17, Lemma 4.1].

Now, let $Q \in \partial\Omega$, $0 < r < \frac{1}{16} \min\{\delta(X^+), \delta(X^-), \text{diam}(\partial\Omega^+)\}$ and let $\tilde{Q} \in \partial\Omega$, \tilde{r} and $V(\tilde{Q}, \tilde{r})$ be as above with $0 < \epsilon \ll 1$ to be chosen. We define two sets

$$B^\pm(\tilde{Q}, \tilde{r}) := \{z \in B(\tilde{Q}, \tilde{r}) \mid \pm \langle z - \tilde{Q}, \hat{n}_V \rangle > \epsilon\tilde{r}\},$$

where \hat{n}_V is a perpendicular to $V(\tilde{Q}, \tilde{r})$. It is well known (see [DJ90]) that if $\partial\Omega$ is Ahlfors regular, then $\partial\Omega^c$ satisfies the corkscrew condition. Thus, for all $0 < \rho < \text{diam}(\partial\Omega^+)$ and $P \in \partial\Omega^+$ there is $Y_{P,\rho} \in B(P, \rho) \cap \partial\Omega^c$ with $\text{dist}(Y_{P,\rho}, \partial\Omega) > \rho/M$, where M depends on AR. Without loss of generality (symmetry of hypothesis), and by choice of ϵ small and choice of the sign of \hat{n}_V , we may assume $A^+ \equiv Y_{\tilde{Q}, \tilde{r}} \in B^+(\tilde{Q}, \tilde{r}) \cap \Omega^+$. Note that $\partial\Omega \cap B^\pm(\tilde{Q}, \tilde{r}) = \emptyset$. In particular, the convexity of $B^+(\tilde{Q}, \tilde{r})$

implies that there cannot be points from both Ω^+ and Ω^- inside $B^+(\tilde{Q}, \tilde{r})$ (similarly for $B^-(\tilde{Q}, \tilde{r})$). We claim $B^-(\tilde{Q}, \tilde{r}) \cap \Omega^- \neq \emptyset$. If this claim is true, then the above argument shows $B^-(\tilde{Q}, \tilde{r}) \subset \Omega^-$ and we set $A^- = \tilde{Q} - 2\varepsilon\tilde{r}\hat{n}_V \in \Omega^- \cap B(\tilde{Q}, \tilde{r})$.

To this end, we appeal to [AMTV16, Lemma 3.3]. In our setting, this lemma yields the following: if $\mathcal{H}^{n+1}(B(\tilde{Q}, \tilde{r}) \cap \Omega^+) \geq \kappa\tilde{r}^{n+1}$, then the doubling of ω^{X^-} implies that $\mathcal{H}^{n+1}(B(\tilde{Q}, \tilde{r}) \cap \Omega^-) \geq \tilde{\kappa}\tilde{r}^{n+1}$, where $\tilde{\kappa} > 0$ depends on C and $\kappa > 0$. Since $B(Y_{\tilde{Q}, \tilde{r}}, \tilde{r}/M) \subset B(\tilde{Q}, \tilde{r}) \cap \Omega^+$, this lemma yields $B(\tilde{Q}, \tilde{r}) \cap \Omega^-$ must intersect $B^+(\tilde{Q}, \tilde{r})$ or $B^-(\tilde{Q}, \tilde{r})$ non-trivially for all ε sufficiently small. Since $B^+(\tilde{Q}, \tilde{r}) \subset \Omega^+$ it must be the case that $B^-(\tilde{Q}, \tilde{r}) \subset \Omega^-$. Having verified our claim, the points A^+, A^- (defined above) suffice as corkscrew points for Q at scale r (we can change location and scale as $B(\tilde{Q}, \tilde{r}) \subset B(Q, r)$ and $\tilde{r} \approx r$). \square

Remark 3.10. We quickly remark that [AMTV16, Lemma 2.1] is used to prove Lemma 3.3 in [AMTV16]. While [AMTV16, Lemma 2.1] is stated for bounded domains, the lemma always holds for open sets with Ahlfors regular boundary (regardless of boundedness).

Our main tool in the proof of Theorem 3.4 is the single layer potential, we recall its definition now:

Definition 3.11 (Riesz transforms and the single layer potential). *Let $E \subset \mathbb{R}^{n+1}$ be an n -dimensional AR (hence closed) set with surface measure σ . We define the (vector valued) Riesz kernel as*

$$(3.12) \quad \mathcal{K}(x) = \tilde{c}_n \frac{x}{|x|^{n+1}}$$

where \tilde{c}_n is chosen so that \mathcal{K} is the gradient of fundamental solution to the Laplacian. For a Borel measurable function f , we then define the Riesz transform

$$(3.13) \quad \mathcal{R}f(X) := \mathcal{K} * (f\sigma)(X) = \int_E \mathcal{K}(X-y)f(y) d\sigma(y) \quad X \in \mathbb{R}^{n+1},$$

as well as the truncated Riesz transforms

$$\mathcal{R}_\varepsilon f(X) := \int_{E \cap \{|X-y|>\varepsilon\}} \mathcal{K}(X-y)f(y) d\sigma(y), \quad \varepsilon > 0.$$

We define \mathcal{S} the single layer potential for the Laplacian relative to E to be

$$(3.14) \quad \mathcal{S}f(X) := \int_E \mathcal{E}(X-y)f(y) d\sigma(y),$$

where $\mathcal{E}(X) = c_n|X|^{1-n}$ is the (positive) fundamental solution to the Laplacian in \mathbb{R}^{n+1} . Notice that $\nabla \mathcal{S}f(X) = \mathcal{R}f(X)$ for $X \notin E$.

Definition 3.15 (Nontangential approach region and maximal function). *Fix $\alpha > 0$ and let Ω be a domain, then for $x \in \partial\Omega$ we define the nontangential approach region (or ‘‘cone’’)*

$$(3.16) \quad \Gamma(x) = \Gamma_\alpha(x) = \{Y \in \Omega : |Y-x| < (1+\alpha)\delta(Y)\}.$$

We also define the nontangential maximal function for $u : \Omega \rightarrow \mathbb{R}$

$$(3.17) \quad \mathcal{N}u(x) = \mathcal{N}_\alpha u(x) = \sup_{Y \in \Gamma_\alpha(x)} |u(Y)|, \quad x \in \partial\Omega.$$

We make the convention that $Nu(x) = 0$ when $\Gamma_\alpha(x) = \emptyset$.

The relationship between the two definitions above is made clear in the following two lemmas:

Lemma 3.18 ([HMT10], [DS91]). *For all $p \in (1, \infty)$ we have*

$$(3.19) \quad \|\mathcal{N}(\nabla \mathcal{S}f)\|_{L^p(d\sigma)} \leq C\|f\|_{L^p(d\sigma)},$$

where C depends on the UR character of $\partial\Omega$, dimension, p , and the aperture of the cones defining \mathcal{N} .

Estimate (3.19) is essentially proved in [DS91]; bounds for the non-tangential maximal function of $\nabla \mathcal{S}f$ follow from uniform bounds for the truncated singular integrals, plus a standard Cotlar Lemma argument; the details may be found in [HMT10, Proposition 3.20].

In addition, we have the following result proved in [HMT10].

Lemma 3.20 ([HMT10] Proposition 3.30). *If Ω is a UR domain (recall Definition 3.3), then for a.e. $x \in \partial\Omega$, and for all $f \in L^p(d\sigma)$, $1 < p < \infty$,*

$$(3.21) \quad \lim_{\substack{Z \rightarrow x \\ Z \in \Gamma^-(x)}} \nabla \mathcal{S}f(Z) = -\frac{1}{2}v(x)f(x) + Tf(x),$$

and

$$(3.22) \quad \lim_{\substack{Z \rightarrow x \\ Z \in \Gamma^+(x)}} \nabla \mathcal{S}f(Z) = \frac{1}{2}v(x)f(x) + Tf(x).$$

where T is a principal value singular integral operator; $\Gamma^+(x)$ is the cone at x relative to Ω , $\Gamma^-(x)$ is the cone at x relative to Ω_{ext} , and v is the outer normal to Ω .

Remark 3.23. As in [BH16], we have taken our fundamental solution to be positive, so for that reason there are some changes in sign in both (3.21) and (3.22) as compared to the formulation in [HMT10].

Now we recall several lemmas from [KT03] and [BH16], most of which require no modification (we indicate the necessary adjustments when this is not the case). The following lemma is a direct result of the John-Nirenberg inequality, which continues to hold for Ahlfors regular sets (see [GaCRdF85, Corollary 2.19, p. 409]).

Lemma 3.24. *Let Ω be a UR domain with $\text{diam}(\partial\Omega) = 1$ and let $0 < r_0 < 1$. Let $f \geq 0$. There exists $\kappa_1 = \kappa_1(n, AR)$ and $C = C(n, AR, r_0)$ such that if $\|\log f\|_*(r_0) < \kappa_1$ then*

$$(3.25) \quad \left(\int_{\Delta} f^4 d\sigma \right)^{\frac{1}{4}} \leq C \int_{\Delta} f d\sigma,$$

for all $\Delta = \Delta(x, r) = B(x, r) \cap \partial\Omega$ with $x \in \partial\Omega$, $r \in (0, 1)$. That is, $f \in RH_4(d\sigma)$.

To prove the Lemma, as in the case of the Euclidean space the John-Nirenberg inequality allows us to show that if $b = \log f$, $e^{\eta b} \in A_2 \subset A_\infty$ where A_2 is the Muckenaupt class where $\eta = \eta(n, AR)$ and $\eta \rightarrow \infty$ as $\|b\|_*(r_0) \rightarrow 0$. Since

$f \in RH_p$ if and only $f^p \in A_\infty$ (see [CUN95]) we then obtain that $f \in RH_4$, provided $\eta > 4$.

Remark 3.26. The reverse Hölder estimate (3.25) (i.e., the A_∞ property) yields an exponential reverse Jensen inequality, so that for any Δ as in (3.25),

$$(3.27) \quad e^{\int_{\Delta} \log f \, d\sigma} \approx \int_{\Delta} f \, d\sigma = \frac{1}{\sigma(\Delta)} \int_{\Delta} f \, d\sigma.$$

See [GaCRdF85, Theorem 2.15, p. 405].

Lemma 3.28. [BH16, Lemma 1.16] *Let Ω be a UR domain with $\text{diam}(\partial\Omega) = 1$. Let $f \geq 0$ with $\|\log f\|_*(r_0) < \kappa_1$, where κ_1 is as in Lemma 3.24. For $\Delta^* := \Delta(x, s) = B(x, s) \cap \partial\Omega$ with $0 \leq s \leq r_0$ and $x \in \partial\Omega$, set*

$$a_{x,s} := e^{\int_{\Delta^*} \log f \, d\sigma}.$$

There exists $C = C(n, AR, r_0)$ such that

$$(3.29) \quad \left(\int_{\Delta^*} \left| 1 - \frac{f}{a} \right|^2 d\sigma \right)^{1/2} \leq C (\|\log f\|_*(r_0))^{1/8},$$

Proof. Following the proof of [BH16, Lemma 1.16] *verbatim*, we replace ϵ with $\|\log f\|_*(r_0)$ and set $p = 2$. We also note that in the last estimate in the proof of [BH16, Lemma 1.16] it is required that $f \in RH_{2p} = RH_4$, at this point we apply Lemma 3.24. \square

If we place additional smallness assumptions on the $\|\cdot\|_*$ -norm of $\log f$ we obtain the additional comparability estimate below. The proof of this lemma is identical to [KT03, Corollary 2.4], appealing to the fact that we may place $f \in RH_p$ for any $p > 1$ provided we force $\log f$ to have small enough $\|\cdot\|_*$ -norm.

Lemma 3.30 ([KT03] Corollary 2.4). *Suppose that Let Ω be a UR domain with $\text{diam}(\partial\Omega) = 1$ and let $0 < r_0 < 1$. There exists $\kappa_2 = \kappa_2(n, AR)$ and $C = C(n, AR, r_0)$ such that if $\|\log f\|_*(r_0) < \kappa_2$ and $\mu(A) = \int_A f \, d\sigma$ then*

$$(3.31) \quad C^{-1} \left(\frac{\sigma(E)}{\sigma(\Delta^*)} \right)^{1+(1/2n)} \leq \frac{\mu(E)}{\mu(\Delta^*)} \leq C \left(\frac{\sigma(E)}{\sigma(\Delta^*)} \right)^{1-(1/2n)},$$

for all surface balls Δ^ and $E \subset \Delta^*$.*

Proof of Theorem 3.4. By scaling we may assume $\text{diam}(\partial\Omega^+) = 1$ and $0 < r_0 < 1$. Our proof will follow the main scheme of [BH16] at first, in fact, our situation is slightly simpler. Suppose first that

$$\|\log k^+\|_*(r_0), \|\log k^-\|_*(r_0) \leq \min\{\kappa_1, \kappa_2\},$$

where κ_1 and κ_2 are the constants from Lemma 3.24 and Lemma 3.28 respectively. By Lemma 3.24, $\omega^{X^+}, \omega^{X^-} \in A_\infty$ and hence ω^{X^+} and ω^{X^-} are doubling measures. From this fact and Lemma 3.7, it follows that Ω^+ and Ω^- satisfy the two-sided (M, R_0) -corkscrew condition with uniform constants depending on $n, UR, r_0, \delta(X^+), \delta(X^-)$.

Let $\|\log k^+\|_*(r_0), \|\log k^-\|_*(r_0) < \eta$ where η is small to be chosen. We first assume that $\eta \ll \min\{r_0, \delta(X^+), \delta(X^-)\} < 1$. Fix $Q \in \partial\Omega$ and $r \in (0, \eta)$ and set

$\Delta = \Delta(Q, r)$. For $y, z \in \Delta$, let y^*, z^* denote arbitrary points in $\Gamma^-(y) \cap B(y, r/2)$ and in $\Gamma^-(z) \cap B(z, r/2)$ respectively. Set $\Delta^* = \Delta(Q, \eta^{-1/(8n)}r)$, where we have chosen η in such a way that the radius of Δ^* is (significantly) smaller than $\min\{r_0, \delta(X^+), \delta(X^-)\}$. As in [BH16], our immediate goal is to show

$$(3.32) \quad \left(\int_{\Delta} \left| \nabla S_{1_{\Delta^*}}(z^*) - \int_{\Delta} \nabla S_{1_{\Delta^*}}(y^*) d\sigma(y) \right|^2 d\sigma(z) \right)^{\frac{1}{2}} \leq C\eta^\gamma,$$

where $\gamma := 1/(8n)$ and $C = C(\delta, n, UR, r_0, \delta(X^+), \delta(X^-))$.

Set $k := k^+$, $a := a_{x, \eta^{-1/(8n)}r} = e^{\int_{\Delta^*} \log k d\sigma}$ and write

$$(3.33) \quad 1_{\Delta^*} = \left[\left(1 - \frac{k}{a}\right) 1_{\Delta^*} \right] + \left[\frac{k}{a} \right] - \left[\left(\frac{k}{a}\right) 1_{(\Delta^*)^c} \right].$$

Using (3.33) we have that the left hand side of (3.32) is bounded by the sum of three terms *I*, *II* and *III* where

$$(3.34) \quad I = \left(\int_{\Delta} \left| \nabla S \left[\left(1 - \frac{k}{a}\right) 1_{\Delta^*} \right] (z^*) - \int_{\Delta} \nabla S \left[\left(1 - \frac{k}{a}\right) 1_{\Delta^*} \right] (y^*) d\sigma(y) \right|^2 d\sigma(z) \right)^{\frac{1}{2}},$$

$$(3.35) \quad II = \left(\int_{\Delta} \left| \nabla S \left[\frac{k}{a} \right] (z^*) - \int_{\Delta} \nabla S \left[\frac{k}{a} \right] (y^*) d\sigma(y) \right|^2 d\sigma(z) \right)^{\frac{1}{2}},$$

and

$$(3.36) \quad III = \left(\int_{\Delta} \left| \nabla S \left[\left(\frac{k}{a}\right) 1_{(\Delta^*)^c} \right] (z^*) - \int_{\Delta} \nabla S \left[\left(\frac{k}{a}\right) 1_{(\Delta^*)^c} \right] (y^*) d\sigma(y) \right|^2 d\sigma(z) \right)^{\frac{1}{2}}.$$

We begin by estimating *I*. By (3.19) and Lemma 3.28, we have

$$(3.37) \quad I \leq 2 \left(\int_{\Delta} \left| \mathcal{N} \left(\nabla S \left[\left(1 - \frac{k}{a}\right) 1_{\Delta^*} \right] \right) \right|^2 d\sigma \right)^{\frac{1}{2}} \\ \lesssim \left(\frac{\sigma(\Delta^*)}{\sigma(\Delta)} \right)^{1/2} \left(\int_{\Delta^*} \left| 1 - \frac{k}{a} \right|^2 d\sigma \right)^{\frac{1}{2}} \lesssim \eta^{\frac{-1}{16}} \eta^{\frac{1}{8}} \lesssim \eta^{\frac{1}{16}}.$$

Now for *II*, we recall that $k = k^{X^+}$ is the Poisson kernel for Ω with pole at X^+ . Moreover, $\mathcal{E}(\cdot - z^*)$ and $\mathcal{E}(\cdot - y^*)$ are harmonic in Ω since $z^*, y^* \in \Omega_{\text{ext}}$, and decay to 0 at infinity, and are therefore equal to their respective Poisson integrals in Ω . Consequently,

$$(3.38) \quad II \leq \frac{1}{a} \left(\int_{\Delta} \int_{\Delta} \left| \nabla \mathcal{E}(X^+ - z^*) - \nabla \mathcal{E}(X^+ - y^*) \right|^2 d\sigma(y) d\sigma(z) \right)^{\frac{1}{2}}.$$

We now apply Lemma 3.30, with $\Delta^* = \Delta(Q, 1) = \partial\Omega$ and $E = \Delta^*$, to deduce that

$$(3.39) \quad \frac{\omega(\partial\Omega)}{\omega(\Delta^*)} = \frac{\omega(\Delta_0^*)}{\omega(\Delta^*)} \lesssim \left(\frac{1}{\eta^{\frac{-1}{8n}} r} \right)^{n+1/2},$$

where $\omega = k^+ d\sigma$. Note that, since $y^*, z^* \in B(x, 2r)$,

$$|\nabla\mathcal{E}(X^+ - z^*) - \nabla\mathcal{E}(X^+ - y^*)| \lesssim \frac{r}{\delta(X^+)^{n+1}} \approx r,$$

where we remind the reader that the implicit constants may depend on $\delta(X^+)$. Then continuing (3.38), we have, using (3.27) and (3.39)

$$\begin{aligned} II &\lesssim \frac{1}{a} r \approx \frac{\sigma(\Delta^*)}{\omega(\Delta^*)} r = \frac{\sigma(\Delta^*)}{\omega(\partial\Omega)} \frac{\omega(\partial\Omega)}{\omega(\Delta^*)} r \\ (3.40) \quad &\lesssim (\eta^{\frac{-1}{8n}} r)^n \left(\frac{1}{\eta^{\frac{-1}{8n}} r} \right)^{n+\frac{1}{2}} r \lesssim \eta^{\frac{1}{16n}} r^{\frac{1}{2}} \lesssim \eta^{\frac{1}{8n}}, \end{aligned}$$

where we have used the estimate $\omega(\partial\Omega) \gtrsim 1$ with implicit constants depending on n AR and $\delta(X^+)$. To see the estimate $\omega(\partial\Omega) \gtrsim 1$, we appeal to Bourgain's estimate [Bou87]. Note that this holds for pole near the boundary, but by using the touching ball for the point X^+ and the Harnack inequality it holds with pole at X^+ , albeit with constants depending on $\delta(X^+)$. Recall the touching ball for a point $X \in \Omega^+$ is the ball $B(X, R)$ with $R = \text{dist}(X, \partial\Omega^+)$.

For III, we use basic Calderón-Zygmund type estimates as follows. Let

$$\Delta'_j := \Delta(Q, 2^j r), \quad A'_j := \Delta'_j \setminus \Delta'_{j-1},$$

so that

$$\begin{aligned} (3.41) \quad III &= \left(\int_{\Delta} \left| \int_{\Delta} \left(\nabla\mathcal{S} \left[\left(\frac{k}{a} \right) 1_{(\Delta^*)^c} \right] (z^*) - \nabla\mathcal{S} \left[\left(\frac{k}{a} \right) 1_{(\Delta^*)^c} \right] (y^*) \right) d\sigma(y) \right|^2 d\sigma(z) \right)^{\frac{1}{2}} \\ &= \left(\int_{\Delta} \left| \int_{\Delta} \int_{\partial\Omega \setminus \Delta^*} \left[\nabla\mathcal{E}(z^* - w) - \nabla\mathcal{E}(y^* - w) \right] \frac{k(w)}{a} d\sigma(w) d\sigma(y) \right|^2 d\sigma(z) \right)^{\frac{1}{2}} \\ &\leq \sum_{\{j | 2^j \geq \eta^{-\frac{1}{8n}}\}} \left(\int_{\Delta} \left[\int_{\Delta} \int_{A'_j} |\nabla\mathcal{E}(z^* - w) - \nabla\mathcal{E}(y^* - w)| \frac{k(w)}{a} d\sigma(w) d\sigma(y) \right]^2 d\sigma(z) \right)^{\frac{1}{2}} \\ &\lesssim \sum_{\{j | 2^j \geq \eta^{-\frac{1}{8n}}\}} \left(\int_{\Delta} \left[\int_{\Delta} \int_{A'_j} \frac{r}{(2^j r)^{n+1}} \frac{k(w)}{a} d\sigma(w) d\sigma(y) \right]^2 d\sigma(z) \right)^{\frac{1}{2}}, \end{aligned}$$

where we understand that the sums are finite and terminate for $2^j r \geq \text{diam}(\partial\Omega) = 1$. We now apply Lemma 3.30, with $\Delta^* = \Delta'_j$ and $E = \Delta^*$, to obtain

$$(3.42) \quad \frac{\omega(\Delta'_j)}{\omega(\Delta^*)} \lesssim \left(\frac{2^j}{\eta^{-\frac{1}{8n}}} \right)^{n+1/2}$$

We then have

$$\begin{aligned}
III &\lesssim \sum_{\{j|2^j \geq \eta^{-\frac{1}{8n}}\}} \frac{1}{2^j} \left(\int_{\Delta} \left[\int_{\Delta} \int_{\Delta'_j} \frac{k(w)}{a} d\sigma(w) d\sigma(y) \right]^2 d\sigma(z) \right)^{\frac{1}{2}} \\
&\lesssim \sum_{\{j|2^j \geq \eta^{-\frac{1}{8n}}\}} \frac{1}{2^j} \frac{1}{a} \frac{\omega(\Delta'_j)}{\sigma(\Delta'_j)} \approx \sum_{\{j|2^j \geq \eta^{-\frac{1}{8n}}\}} \frac{1}{2^j} \frac{\sigma(\Delta^*)}{\sigma(\Delta'_j)} \frac{\omega(\Delta'_j)}{\omega(\Delta^*)} \\
&\lesssim \sum_{\{j|2^j \geq \eta^{-\frac{1}{8n}}\}} 2^{-j} \left(\frac{\eta^{-\frac{1}{8n}}}{2^j} \right)^n \left(\frac{2^j}{\eta^{-\frac{1}{8n}}} \right)^{n+1/2} \\
&\lesssim \eta^{\frac{1}{16n}} \sum_{\{j|2^j \geq \eta^{-\frac{1}{8n}}\}} 2^{-j/2} \lesssim \eta^{\frac{1}{8n}},
\end{aligned}$$

where in the second line we have used (3.27) and in the second to last line the AR property and (3.42). Combining the estimates for I, II, III , we obtain (3.32).

Setting

$$n.t.\nabla S^- f(x) := \lim_{\substack{Z \rightarrow x \\ Z \in \Gamma^-(x)}} \nabla S f(Z),$$

since the limit exists for a.e. $x \in \partial\Omega$ (see Lemma 3.20), we may now use (3.19), (3.32), and dominated convergence to obtain

$$(3.43) \quad \left(\int_{\Delta} \left| n.t.\nabla S^- 1_{\Delta^*}(z) - \int_{\Delta} n.t.\nabla S^- 1_{\Delta^*}(y) d\sigma(y) \right|^2 d\sigma(z) \right)^{\frac{1}{2}} \leq C\eta^\gamma.$$

In addition, since $\|\log k^-\|_*(r_0) < \eta$, the same analysis shows that (3.43) holds for $n.t.\nabla S^- 1_{\Delta^*}$ replaced with

$$(3.44) \quad n.t.\nabla S^+ 1_{\Delta^*} := \lim_{\substack{Z \rightarrow x \\ Z \in \Gamma^+(x)}} \nabla S 1_{\Delta^*}(Z).$$

By (3.21) and (3.22)

$$(3.45) \quad v(x) 1_{\Delta^*}(x) = \lim_{\substack{Z \rightarrow x \\ Z \in \Gamma^+(x)}} \nabla S 1_{\Delta^*}(Z) - \lim_{\substack{Z \rightarrow x \\ Z \in \Gamma^-(x)}} \nabla S 1_{\Delta^*}(Z).$$

Thus, since $\Delta \subset \Delta^*$, by (3.43) and its analogue for S^+ , we obtain for all $\Delta = \Delta(x, r)$ with $x \in \partial\Omega^+$ and $r \in (0, \eta)$

$$(3.46) \quad \left(\int_{\Delta} \left| v(z) - \int_{\Delta} v(y) d\sigma(y) \right|^2 d\sigma(z) \right)^{\frac{1}{2}} \leq C\eta^\gamma.$$

Now, we depart from [BH16] and apply a variant of our compactness argument in Theorem 2.1. Notice that, writing (3.46) compactly, we have

$$\|v\|_*(Q, \eta) < C\eta^\gamma,$$

where $\|v\|_*(Q, \eta)$ is as in Corollary 2.20. We remind the reader that Ω^+ satisfies the two-sided (M, R_0) -corkscrew condition with constants independent of η . Choosing η such that $C\eta^\gamma < c_\delta$ we apply the ideas of Corollary 2.20 to obtain that $\partial\Omega$ is $(\delta, \eta r_\delta)$ -Reifenberg flat (and thus a $(\delta, \eta r_\delta)$ -chord arc domain). \square

We close with a final modification to Theorems 1.1 and 1.2 in [BH16]. In the presence of an additional hypothesis to ensure the existence of corkscrews, then in the setting of Theorems 1.1 and 1.2 in [BH16], we may also conclude that $\partial\Omega$ is vanishing Reifenberg flat. This follows immediately from Corollary 2.22. In fact, using Lemma 3.7, an additional hypothesis is not required except for the situation that $\partial\Omega$ has infinite diameter and both poles are finite. Indeed, we had this application in mind when we began working on this paper.

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REFERENCES

- [AHM⁺17] Jonas Azzam, Steve Hofmann, José Marí a Martell, Kaj Nyström, and Tatiana Toro, *A new characterization of chord-arc domains*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 4, 967–981. MR 3626548 [17](#)
- [AMTV16] Jonas Azzam, Mihalis Mourgoglou, Xavier Tolsa, and Alexander Volberg, *On a two-phase problem for harmonic measure in general domains*, preprint, arXiv:1609.06133 (2016). [17](#), [18](#)
- [BET17] Matthew Badger, Max Engelstein, and Tatiana Toro, *Structure of sets which are well approximated by zero sets of harmonic polynomials*, Analysis and PDE **10** (2017), no. 6, 1455–1495. [6](#)
- [BH16] Simon Bortz and Steve Hofmann, *A singular integral approach to a two phase free boundary problem*, Proc. Amer. Math. Soc. **144** (2016), no. 9, 3959–3973. MR 3513552 [2](#), [5](#), [13](#), [15](#), [16](#), [17](#), [19](#), [20](#), [21](#), [23](#), [24](#)
- [Bou87] J. Bourgain, *On the Hausdorff dimension of harmonic measure in higher dimension*, Invent. Math. **87** (1987), no. 3, 477–483. MR 874032 [22](#)
- [CFMS81] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, *Boundary behavior of nonnegative solutions of elliptic operators in divergence form.*, Indiana Univ. Math. J. **30** (1981), 621–640 (English). [17](#)
- [CUN95] David Cruz-Uribe and C. J. Neugebauer, *The structure of the reverse Hölder classes*, Trans. Amer. Math. Soc. **347** (1995), no. 8, 2941–2960. MR 1308005 [20](#)
- [CW71] Ronald R. Coifman and Guido Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Mathematics, Vol. 242, Springer-Verlag, Berlin-New York, 1971, Étude de certaines intégrales singulières. MR 0499948 [5](#)
- [DJ90] G. David and D. Jerison, *Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals*, Indiana Univ. Math. J. **39** (1990), no. 3, 831–845. MR 1078740 [3](#), [16](#), [17](#)
- [DS91] G. David and S. Semmes, *Singular integrals and rectifiable sets in \mathbf{R}^n : Beyond Lipschitz graphs*, Astérisque (1991), no. 193, 152. MR 1113517 [15](#), [19](#)
- [DS93] ———, *Analysis of and on uniformly rectifiable sets*, Mathematical Surveys and Monographs, vol. 38, American Mathematical Society, Providence, RI, 1993. MR 1251061 (94i:28003) [15](#), [17](#)
- [EG92] Lawrence C. Evans and Ronald F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992. MR 1158660 [7](#), [8](#), [9](#), [16](#)
- [GaCRdF85] José Garcá a Cuerva and José L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985, Notas de Matemática [Mathematical Notes], 104. MR 807149 [19](#), [20](#)

- [HMT10] Steve Hofmann, Marius Mitrea, and Michael Taylor, *Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro.*, Int. Math. Res. Not. **2010** (2010), no. 14, 2567–2865 (English). [2](#), [5](#), [6](#), [15](#), [16](#), [19](#)
- [JK82] David S. Jerison and Carlos E. Kenig, *Boundary behavior of harmonic functions in nontangentially accessible domains*, Adv. in Math. **46** (1982), no. 1, 80–147. MR 676988 (84d:31005b) [3](#)
- [KT97] Carlos E. Kenig and Tatiana Toro, *Harmonic measure on locally flat domains*, Duke Math. J. **87** (1997), no. 3, 509–551. MR 1446617 (98k:31010) [2](#), [3](#), [4](#), [13](#), [17](#)
- [KT99] ———, *Free boundary regularity for harmonic measures and Poisson kernels*, Ann. of Math. (2) **150** (1999), no. 2, 369–454. MR 1726699 (2001d:31004) [2](#), [5](#)
- [KT03] ———, *Poisson kernel characterization of Reifenberg flat chord arc domains*, Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 3, 323–401. MR 1977823 (2004d:31010) [2](#), [5](#), [6](#), [12](#), [17](#), [19](#), [20](#)
- [KT06] ———, *Free boundary regularity below the continuous threshold: 2-phase problems*, J. Reine Angew. Math. **596** (2006), 1–44. MR 2254803 (2007k:35526) [1](#), [2](#), [3](#), [15](#)
- [Mer16a] Jessica Merhej, *On the geometry of rectifiable sets with carleson and poincaré-type conditions*, preprint, arXiv:1510.05056. To appear in Indiana Univ. Math. J., 2016. [2](#)
- [Mer16b] ———, *Poincaré-type inequalities and finding good parameterizations*, preprint, arXiv:1605.07655., 2016. [2](#)
- [MMV96] Pertti Mattila, Mark S. Melnikov, and Joan Verdera, *The Cauchy integral, analytic capacity, and uniform rectifiability.*, Ann. Math. (2) **144** (1996), no. 1, 127–136 (English). [15](#)
- [MPT14] Emmanouil Milakis, Jill Pipher, and Tatiana Toro, *Perturbations of elliptic operators in chord arc domains*, Harmonic analysis and partial differential equations, Contemp. Math., vol. 612, Amer. Math. Soc., Providence, RI, 2014, pp. 143–161. MR 3204862 [2](#)
- [NTV14] Fedor Nazarov, Xavier Tolsa, and Alexander Volberg, *On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1.*, Acta Math. **213** (2014), no. 2, 237–321 (English). [15](#)
- [NV17] Aaron Naber and Daniele Valtorta, *Rectifiable-Reifenberg and the regularity of stationary and minimizing harmonic maps*, Ann. of Math. (2) **185** (2017), no. 1, 131–227. MR 3583353 [2](#)
- [Rei60] E. R. Reifenberg, *Solution of the Plateau Problem for m -dimensional surfaces of varying topological type*, Acta Math. **104** (1960), 1–92. MR 0114145 (22 #4972) [1](#), [2](#), [3](#)
- [Sem91a] Stephen Semmes, *Chord-arc surfaces with small constant. I.*, Adv. Math. **85** (1991), no. 2, 198–223 (English). [1](#), [2](#)
- [Sem91b] ———, *Chord-arc surfaces with small constant. II: Good parametrizations.*, Adv. Math. **88** (1991), no. 2, 170–199 (English). [1](#), [2](#)
- [Tor95] Tatiana Toro, *Geometric conditions and existence of bi-Lipschitz parameterizations*, Duke Math. J. **77** (1995), no. 1, 193–227. MR 1317632 [2](#)
- [Tor97] ———, *Doubling and flatness: geometry of measures*, Notices Amer. Math. Soc. **44** (1997), no. 9, 1087–1094. MR 1470167 [1](#)

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