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**Math 4567. Midterm Exam III (take home) Solutions**

**April 23, 2010**

There are a total of 100 points and 6 problems on this take home exam.

| Problem | Score |
|---|---|
| 1. | |
| 2. | |
| 3. | |
| 4. | |
| 5. | |
| 6. | |
| Total: | |

1. **Chapter 5, page 113, Problem 2. (20 points).** A solid body 40 cm in diameter, initially at $100^o$ C throughout, is cooled by keeping its surface at $0^o$ C. Use the temperature formula for

$$u(r,t) = \frac{1}{r}\sum_{n=1}^{\infty} B_n \exp(-\frac{n^2\pi^2 k}{a^2}t)\sin\frac{n\pi r}{a}, \;\; B_n = \frac{2}{a}\int_0^a rf(r)\sin\frac{n\pi r}{a}\, dr,$$

to show formally that

$$u(0+,t) = 200\sum_{n=1}^{\infty}(-1)^{n+1}\exp(-\frac{n^2\pi^2 k}{400}t).$$

Find the approximate temperature at the center of the sphere 10 min after cooling begins if (a) $k = 0.15$ cgs unit ; and (b) $k = 0.005$ cgs unit. Make sure that your answer is accurate to within 1/10th of a degree Celsius and justify your reasoning.

**Solution**: Here $f(r) = 100^o$ and $a = 20$ cm. Thus

$$B_n = \frac{1}{10}\int_0^{20} rf(r)\sin\frac{n\pi r}{20}\, dr =$$

$$\frac{10}{n\pi}\{-r\cos\frac{n\pi r}{20}|_0^{20} + \int_0^{20}\cos\frac{n\pi r}{20}dr\} = \frac{200}{n\pi}(-1)^{n+1}.$$

We use the limit

$$\lim_{x\to 0+}\frac{\sin x}{x} = 1, \;\text{ or }\; \lim_{x\to 0+}\frac{\sin Bx}{x} = B\lim_{Bx\to 0+}\frac{\sin Bx}{Bx} = B$$

Thus

$$\lim_{r\to 0+}\frac{\sin\frac{n\pi r}{a}}{r} = \frac{n\pi}{a},$$

It follows that

$$u(0+,t) = 200\sum_{n=1}^{\infty}(-1)^{n+1}\exp(-\frac{n^2\pi^2 k}{400}t).$$

After 10 minutes=600 seconds

$$u(0+,600) = 200\sum_{n=1}^{\infty}(-1)^{n+1}\exp(-3n^2\pi^2 k/2).$$

This is a convergent alternating series with monotonically decreasing terms. Thus the partial sum of an odd number of terms will be too large and the sum of an even number of terms will be too small, with the difference between the odd and even sums decreasing monotonically to 0.

(a) If $k = 0.15$ then taking only the first term in the series gives 21.71 (to 2 decimal place accuracy). Taking the sum of the first two terms gives the lower bound 21.68. Thus the temperature is $21.7^o$ C to the nearest tenth of a degree;

(b) If $k = 0.005$ we need to sum the first 10 terms to give a lower bound of 99.98 (to 2 decimal place accuracy). Summing the first 11 terms gives an upper bound 100.01. Thus the temperature is $100^o$ C to the nearest tenth of a degree.

2. **Chapter 5, page 117, Problem 1. (20 points)** Solve the boundary value problem

$$u_t = u_{xx} + xp(t), \quad 0 < x < 1,\ t > 0,$$

$$u(0,t) = 0,\ u(1,t) = 0,\ u(x,0) = 0.$$

Use the expansion

$$x = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x, \quad 0 < x < 1,$$

to get

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x \int_0^t e^{-n^2\pi^2(t-\tau)} p(\tau)d\tau.$$

Why does this series converge?Why does this series converge?

**Solution**: Write

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin n\pi x \ dx,$$

so that

$$B_n(t) = 2\int_0^1 u(x,t) \sin n\pi x \ dx, \quad n = 1, 2, \cdots.$$

Then

$$B_n'(t) = 2\int_0^1 u_t(x,t) \sin n\pi x \ dx = 2\int_0^1 (u_{xx}(x,t) + xp(t)) \sin n\pi x \ dx$$

$$= 2p(t)\int_0^1 x \sin n\pi x \ dx + 2\int_0^1 u_{xx}(x,t) \sin n\pi x \ dx =$$

$$2p(t)\frac{(-1)^{n+1}}{n\pi} - 2n^2\pi^2 \int_0^1 u(x,t) \sin n\pi x \ dx,$$

where we have integrated by parts twice and made use of the boundary conditions. Thus

$$B_n'(t) = 2p(t)\frac{(-1)^{n+1}}{n\pi} - n^2\pi^2 B_n(t).$$

The integrating factor for this first order linear differential equation is $e^{n^2\pi^2 t}$ so we have

$$\frac{d}{dt}(e^{n^2\pi^2 t}B_n(t)) = 2p(t)\frac{(-1)^{n+1}}{n\pi}e^{n^2\pi^2 t}.$$

Thus

$$B_n(t) = b_n e^{-n^2\pi^2 t} + \frac{2(-1)^{n+1}}{n\pi}\int_0^t p(\tau)e^{n^2\pi^2(\tau-t)}d\tau.$$

Now

$$u(x,0) = 0 = \sum_{n=1}^{\infty} b_n \sin n\pi x,$$

so $b_n = 0$. Thus

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x\int_0^t e^{-n^2\pi^2(t-\tau)}p(\tau)d\tau.$$

To check convergence, let $K = \max_{t\in[0,M]}|p(t)|$ and let

$$C_n(t) = \frac{2}{\pi}\frac{(-1)^{n+1}}{n}\sin n\pi x\int_0^t e^{-n^2\pi^2(t-\tau)}p(\tau)d\tau.$$

Then

$$|C_n(t)| \le \frac{2}{\pi n}e^{-n^2\pi^2 t}\int_0^M e^{n^2\pi^2\tau}K\ d\tau = \frac{2K}{\pi^3 n^3}e^{-n^2\pi^2 t}(e^{n^2\pi^2 M} - 1)$$

for $t > M$, and

$$|C_n(t)| \le \frac{2}{\pi n}e^{-n^2\pi^2 t}\int_0^t e^{n^2\pi^2\tau}K\ d\tau = \frac{2K}{\pi^3 n^3}(1 - e^{-n^2\pi^2 t})$$

for $0 < t \le M$. Thus for all $t \ge 0$ there is a constant $H > 0$ such that $|C_n(t)| \le H/n^3$. The series for $u(x,t)$ converges absolutely and uniformly.

3. **Chapter 8, page 215, problem 6. (10 points)** Use the normalized eigenfunctions in Problem 2, page 209 to derive

$$x\left(\frac{2+h}{1+h}-x\right)=4h\sum_{n=1}^{\infty}\frac{1-\cos\alpha_n}{\alpha_n^3(h+\cos^2\alpha_n)}\sin\alpha_n x,\quad 0<x<1,$$

where $\tan\alpha_n=-\alpha_n/h$, $\alpha>0$.

**Solution**:

$$x(\frac{2+h}{1+h}-x)=\sum_{n=1}^{\infty}c_n\phi_n(x),\; c_n=\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\int_0^1 s(\frac{2+h}{1+h}-s)\sin\alpha_n s\,ds.$$

Integrating by parts twice we find

$$c_n=\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\left\{-\frac{\cos\alpha_n}{(1+h)\alpha_n}-\frac{h\sin\alpha_n}{(1+h)\alpha_n^2}+2\frac{(1-\cos\alpha_n)}{\alpha_n^3}\right\}.$$

Then, making use of the identity $-h\sin\alpha_n=\alpha_n\cos\alpha_n$, we find

$$c_n\sqrt{\frac{2h}{h+\cos^2\alpha_n}}=\frac{2h}{\alpha_n^3(h+\cos^2\alpha_n)}\left[-\frac{\alpha_n^2\cos\alpha_n}{1+h}-\frac{h\alpha_n\sin\alpha_n}{1+h}+2(1-\cos\alpha_n)\right]$$

$$=\frac{4h(1-\cos\alpha_n)}{\alpha_n^3(h+\cos^2\alpha_n)},$$

which yields the desired result.

4. **Chapter 8, page 215, problem 7.** (10 points) Use the normalized eigenfunctions in Problem 1, page 209 to derive

$$\sin \omega x = 2\omega \cos \omega \sum_{n=1}^{\infty} \frac{(-1)^n}{\omega^2 - \omega_n^2} \sin \omega_n x, \quad 0 < x < 1,$$

where

$$\omega_n = \frac{(2n-1)\pi}{2}, \text{ and } \omega \neq \omega_n, \text{ for any n.}$$

**Solution**: The normalized eigenfunctions are

$$\phi_n(x) = \sqrt{2} \sin \omega_n x, \ \omega_n = \frac{(2n-1)\pi}{2}, \quad n = 1, 2, \cdots$$

and $0 < x < 1$. Thus

$$\sin \omega x = \sum_{n=1}^{\infty} C_n \phi_n(x), \ 0 < x < 1, \quad C_n = \int_0^1 \sin \omega x \ \phi_n(x) dx.$$

Thus

$$C_n = \sqrt{2} \int_0^1 \sin \omega x \sin \omega_n x \ dx.$$

Since

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B).$$

we have

$$2 \sin \omega x \sin \omega_n x = \cos(\omega - \omega_n)x - \cos(\omega + \omega_n)x,$$

so

$$C_n = \frac{1}{\sqrt{2}} \left[ \frac{\sin(\omega - \omega_n)x}{\omega - \omega_n} |_0^1 - \frac{\sin(\omega + \omega_n)x}{\omega + \omega_n} |_0^1 \right].$$
$$= \frac{1}{\sqrt{2}} \left( \frac{\sin(\omega - \omega_n)}{\omega - \omega_n} - \frac{\sin(\omega + \omega_n)}{\omega + \omega_n} \right)$$

. But $\cos \omega_n = 0$, $\sin \omega_n = (-1)^{n+1}$. Thus

$$C_n = \frac{1}{\sqrt{2}} \left( \frac{\sin \omega \cos \omega_n - \cos \omega \sin \omega_n}{\omega - \omega_n} - \frac{\sin \omega \cos \omega_n + \cos \omega \sin \omega_n}{\omega + \omega_n} \right)$$
$$= \frac{(-1)^n}{\sqrt{2}} \left( \frac{1}{\omega - \omega_n} + \frac{1}{\omega + \omega_n} \right) \cos \omega$$
$$= (-1)^n \sqrt{2} \frac{\omega \cos \omega}{\omega^2 - \omega_n^2}.$$

The result follows immediately from this.

5. **Chapter 6, page 157, Problem 3. (20 points)**

**(a)** Show that the function

$$f(x) = \begin{cases} 0 & \text{when } x < 0, \\ \exp(-x) & \text{when } x > 0, \\ \frac{1}{2} & \text{when } x = 0, \end{cases}$$

satisfies the conditions of the Fourier integral pointwise convergence theorem. Establish

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos \alpha x + \alpha \sin \alpha x}{1 + \alpha^2} \, d\alpha, \quad -\infty < x < \infty.$$

**(b)** Verify this directly at the point $x = 0$.

**Solution**:

**(a)** $f$ is piecewise continuous on every bounded interval and

$$\int_{-\infty}^\infty |f(x)| \, dx = \int_0^\infty \exp(-x) \, dx = 1 < \infty,$$

so

$$\frac{f(x+) + f(x-)}{2} = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(s) \cos \alpha(s - x) \, ds \, d\alpha,$$

at each $x$ such that $f'_R(x)$ and $f'_L(x)$ exist, and these derivatives exist at all $x$. Further, this function satisfies

$$\frac{f(x+) + f(x-)}{2} = f(x)$$

for all $x$. Now

$$\int_{-\infty}^\infty f(s) \cos \alpha(s - x) \, ds = \int_0^\infty \exp(-s) \cos \alpha(s - x) \, ds,$$

and, integrating by parts twice,

$$\int_0^\infty \exp(-s) \cos \alpha(s-x) ds = \cos \alpha x + \alpha \sin \alpha x - \alpha^2 \int_0^\infty \exp(-s) \cos \alpha(s-x) ds,$$

so

$$\int_0^\infty \exp(-s) \cos \alpha(s - x) ds = \frac{\cos \alpha x + \alpha \sin \alpha x}{1 + \alpha^2}.$$

Thus

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos \alpha x + \alpha \sin \alpha x}{1 + \alpha^2} \, d\alpha.$$

**(b)** In the special case $x = 0$ we can evaluate the integral directly to get

$$\frac{1}{\pi}\int_0^\infty \frac{1}{1+\alpha^2}\,d\alpha = \frac{1}{\pi}\arctan\alpha\Big|_0^\infty = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2} = f(0).$$

6. **(20 points)** Use the real form of the Fourier transform pair for the real-valued function $f(x)$,

$$f(x) = \int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\ d\alpha,$$

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\cos\alpha t\ dt, \quad B(\alpha) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\sin\alpha t\ dt,$$

with Parseval formula

$$\frac{1}{\pi}\int_{-\infty}^\infty f^2(x)\ dx = \int_0^\infty \left(A^2(\alpha) + B^2(\alpha)\right) d\alpha,$$

To derive the complex form of the transform pair for $f(x)$:

$$f(x) = \int_{-\infty}^\infty \hat{f}(\lambda)e^{i\lambda x}d\lambda,$$

$$\hat{f}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{-i\lambda x}dx,$$

$$\frac{1}{2\pi}\int_{-\infty}^\infty f^2(x)\ dx = \int_{-\infty}^\infty |\hat{f}(\lambda)|^2\ d\lambda.$$

The similar computation relating real and complex forms of the Fourier series in problem 8, Chapter 2, page 42 should prove helpful. How would the formulas change if $f(x)$ was a complex valued function?

**Solution**: We define

$$\hat{f}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{-i\lambda x}dx = \frac{1}{2}\left(A(\alpha) - iB(\alpha)\right)$$

for $\lambda = \alpha > 0$ and

$$\hat{f}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{-i\lambda x}dx = \frac{1}{2}\left(A(\alpha) + iB(\alpha)\right)$$

for $\lambda = -\alpha < 0$. Then

$$\int_{-\infty}^\infty \hat{f}(\lambda)e^{i\lambda x}d\lambda =$$

$$\int_0^\infty \frac{1}{2}\left(A(\alpha) + iB(\alpha)\right)e^{-i\alpha x}d\alpha + \int_0^\infty \frac{1}{2}\left(A(\alpha) - iB(\alpha)\right)e^{i\alpha x}d\alpha$$

$$= \int_0^{\infty} [A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x] \; d\alpha = f(x).$$

Similarly,

$$\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 \; d\lambda. =$$

$$\int_0^{\infty} [\frac{1}{2} (A(\alpha) + iB(\alpha)) \frac{1}{2} (A(\alpha) - iB(\alpha))] d\alpha$$

$$\int_0^{\infty} [\frac{1}{2} (A(\alpha) - iB(\alpha)) \frac{1}{2} (A(\alpha) + iB(\alpha))] d\alpha$$

$$= \frac{1}{2} \int_0^{\infty} [A^2(\alpha) + B^2(\alpha)] d\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^2(x) \; dx.$$

If $f(x)$ is a complex valued function, then the squares in the formulas above are replaced by absolute values squared:

$$f(x) = \int_0^{\infty} [A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x] \; d\alpha,$$

$$A(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \alpha t \; dt, \quad B(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \alpha t \; dt,$$

with Parseval formula

$$\frac{1}{\pi} \int_{-\infty}^{\infty} |f(x)|^2 \; dx = \int_0^{\infty} \left( |A(\alpha)|^2 + |B(\alpha)|^2 \right) d\alpha,$$

and

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} d\lambda,$$

$$\hat{f}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx,$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2 \; dx = \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 \; d\lambda.$$