Positivity for cluster algebras from surfaces

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(Joint work with Ralf Schiffler (University of Connecticut)
and Lauren Williams (University of California, Berkeley))

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http//math.mit.edu/∼musiker/ClusterSurface.pdf

Outline

1. Introduction: the Laurent phenomenon, and the positivity conjecture of Fomin-Zelevinsky.

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2. Cluster algebras arising from triangulated surfaces (following Fomin-Shapiro-Thurston's terminology).
3. Graph theoretic construction for surfaces with or without punctures (joint work with Schiffler and Williams).
4. Examples of this construction.

Introduction to Cluster Algebras

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Cluster algebras are a certain class of commutative rings which have a distinguished set of generators that are grouped into overlapping subsets, called clusters, each having the same cardinality.

What is a Cluster Algebra?

Definition (Sergey Fomin and Andrei Zelevinsky 2001) A cluster algebra $\mathcal{A}$ (of geometric type) is a subalgebra of $k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ constructed cluster by cluster by certain exchange relations.

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Relations:

Induced by the Binomial Exchange Relations.

Example: Coordinate Ring of Grassmannian$(2, n+3)$

Let $Gr_{2,n+3} = \{V | V \subset \mathbb{C}^{n+3},\ \dim V = 2\}$ planes in $(n+3)$-space

Elements of $Gr_{2,n+3}$ represented by 2-by-$(n+3)$ matrices of full rank.

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Plücker coordinates $p_{ij}(M) =$ det of 2-by-2 submatrices in columns i and j.

The coordinate ring $\mathbb{C}[Gr_{2,n+3}]$ is generated by all the p_{ij}'s for $1 \leq i < j \leq n+3$ subject to the Plücker relations given by the 4-tuples

$$p_{ik}p_{j\ell} = p_{ij}p_{k\ell} + p_{i\ell}p_{jk} \text{ for } i < j < k < \ell.$$

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Claim. $\mathbb{C}[Gr_{2,n+3}]$ has the structure of a cluster algebra. Clusters are maximal algebraically independent sets of p_{ij}'s.

Each have size $(2n+3)$ where $(n+3)$ of the variables are frozen and n of them are exchangeable.

Example: Coordinate Ring of Grassmannian$(2, n+3)$

Cluster algebra structure of $Gr_{2,n+3}$ as a triangulated $(n+3)$-gon.

Frozen Variables / Coefficients $\longleftrightarrow$ sides of the $(n+3)$-gon

Cluster Variables $\longleftrightarrow$ $\{p_{ij} : |i-j| \neq 1 \mod (n+3)\}$ $\longleftrightarrow$ diagonals

Seeds $\longleftrightarrow$ triangulations of the $(n+3)$-gon

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Clusters $\longleftrightarrow$ Set of p_{ij}'s corresponding to a triangulation

Can exchange between various clusters by flipping between triangulations.

This is called mutation, and we will present a detailed example later.

Another Example: Total Positivity

Given a 3-by-3 matrix $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in SL_3$, how do you check whether it is totally positive, meaning that all minors are positive?

(i.e. $a > 0,\ b > 0,\ c > 0, \ldots, ae - bd > 0, \ldots, \det M > 0$.)

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Answer: It is sufficient to check that $c > 0$, $g > 0$, $bf - ce > 0$, $dh - eg > 0$ and four other conditions

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There are exactly 50 such overlapping sets of four conditions. These 50 algebraic elements generate a cluster algebra structure (i.e. binomial exchange relations among the elements).

Seeds and Mutation

Definition. A Seed is a pair $(\mathbb{X}, B)$, where $\mathbb{X} = \{x_1, x_2, \ldots, x_{n+m}\}$ is an initial Cluster, and B is an Exchange Matrix, i.e. a $(n+m)$-by-n skew-symmetrizable integral matrix. $(d_i b_{ij} = -d_j b_{ji}$ for $d_i \in \mathbb{Z}_{>0})$

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Columns of B encode the exchanges

$$x_k x_k' = \prod_{b_{ik}>0} x_i^{|b_{ik}|} + \prod_{b_{ik}<0} x_i^{|b_{ik}|}$$

for $k \in \{1, 2, \dots n\}$. Note: If only one sign occurs (e.g. $b_{ik} > 0$), we get a monomial of 1.

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for $k \in \{1, 2, \dots n\}$. Note: If only one sign occurs (e.g. $b_{ik} > 0$), we get a monomial of 1. For all $k \in \{1, 2, \dots, n\}$, there exists another seed consisting of cluster $\{x_1, \dots, \widehat{x_k}, \dots, x_{n+m}\} \cup \{x_k'\}$ and matrix $\mu_k(B)$.

$$\mu_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if} \quad k = i \quad \text{or} \quad k = j \\ b_{ij} & \text{if} \quad b_{ik} b_{kj} \leq 0 \\ b_{ij} + b_{ik} b_{kj} & \text{if} \quad b_{ik}, b_{kj} > 0 \\ b_{ij} - b_{ik} b_{kj} & \text{if} \quad b_{ik}, b_{kj} < 0 \end{cases}$$

Finite Mutation Type and Finite Type

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Theorem. (FZ 2002) Finite type cluster algebras can be described via the Cartan-Killing classification of Lie algebras.

Example 2: Rank 2 Cluster Algebras

Let $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$, $b, c \in \mathbb{Z}_{>0}$. $(\{x_1, x_2\}, B)$ is a seed for a cluster algebra of rank 2.

$$\mu_1(B) = \mu_2(B) = -B \ \text{ and } \ x_1 x_1' = x_2^c + 1, \quad x_2 x_2' = 1 + x_1^b.$$

Thus the cluster variables in this case are

$$\{x_n : n \in \mathbb{Z}\} \text{ satisfying } x_n x_{n-2} = \begin{cases} x_{n-1}^b + 1 \text{ if } n \text{ is odd} \\ x_{n-1}^c + 1 \text{ if } n \text{ is even} \end{cases}.$$

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Example 3. Let $\mathcal{A}$ be the cluster algebra defined by the initial cluster $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ and the initial exchange pattern

$$x_1 x_1' = y_1 + x_2, \quad x_2 x_2' = x_1 x_3 y_2 + 1, \quad x_3 x_3' = y_3 + x_2. \quad \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$\mathcal{A}$ is of finite type, type A_3 and corresponds to a triangulated hexagon.

$$\left\{ x_1, x_2, x_3, \frac{y_1 + x_2}{x_1}, \frac{x_1 x_3 y_2 + 1}{x_2}, \frac{y_3 + x_2}{x_3}, \frac{x_1 x_3 y_1 y_2 + y_1 + x_2}{x_1 x_2}, \right.$$
$$\left. \frac{x_1 x_3 y_2 y_3 + y_3 + x_2}{x_2 x_3}, \frac{x_1 x_3 y_1 y_2 y_3 + y_1 y_3 + x_2 y_3 + x_2 y_1 + x_2^2}{x_1 x_2 x_3} \right\}.$$

The y_i's are known as principal coefficients.

The Positivity Conjecture of Fomin and Zelevinsky

Theorem. (The Laurent Phenomenon FZ 2001) For any cluster algebra defined by initial seed $(\{x_1, x_2, \ldots, x_{n+m}\}, B)$, all cluster variables of $\mathcal{A}(B)$ are Laurent polynomials in $\{x_1, x_2, \ldots, x_{n+m}\}$

(with no coefficient $x_{n+1}, \ldots, x_{n+m}$ in the denominator).

Because of the Laurent Phenomenon, any cluster variable x_α can be expressed as $\frac{P_\alpha(x_1,\ldots,x_{n+m})}{x_1^{\alpha_1}\cdots x_n^{\alpha_n}}$ where $P_\alpha \in \mathbb{Z}[x_1, \ldots, x_{n+m}]$ and the α_i's $\in \mathbb{Z}$.

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Conjecture. (FZ 2001) For any cluster variable x_α and any initial seed (i.e. initial cluster $\{x_1, \ldots, x_{n+m}\}$ and initial exchange pattern B), the polynomial $P_\alpha(x_1, \ldots, x_n)$ has nonnegative integer coefficients.

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Work of [Carroll-Price 2002] gave expansion formulas for case of Ptolemy algebras, cluster algebras of type A_n with boundary coefficients $(Gr_{2,n+3})$.

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Main Theorem

Theorem. (Positivity for cluster algebras from surfaces MSW 2009)
Let $\mathcal{A}$ be any cluster algebra arising from a surface (with or without punctures), where the coefficient system is of geometric type, and let Σ be any initial seed.

Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

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Due to work of Felikson-Shapiro-Tumarkin, we get

Corollary. Positivity for any seed, for all but 11 skew-symmetric cluster algebras of finite mutation type. (Rank two skew-symmetric cases by Caldero-Reineke)

Cluster Algebras of Triangulated Surfaces

We follow (Fomin-Shapiro-Thurston), based on earlier work of Fock-Goncharov and Gekhtman-Shapiro-Vainshtein.

We have a surface S with a set of marked points M. (If $P \in M$ is in the interior of S, i.e. $S \setminus \delta S$, then P is known an a puncture).

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An arc γ satisfies (we care about arcs up to isotopy)

1. The endpoints of γ are in M.
2. γ does not cross itself.
3. except for the endpoints, γ is disjoint from M and the boundary of S.
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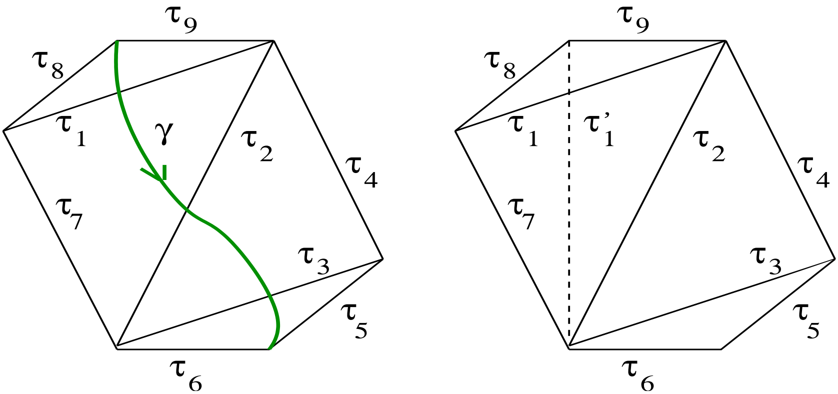
$$\text{Seed} \leftrightarrow \text{Triangulation } T = \{\tau_1, \tau_2, \ldots, \tau_n\}$$

$$\text{Cluster Variable} \leftrightarrow \text{Arc } \gamma \ (x_i \leftrightarrow \tau_i \in T)$$

$$\text{Cluster Mutation} \leftrightarrow \text{Ptolemy Exchanges (Flipping Diagonals).}$$

Example of Hexagon

Consider the triangulated hexagon (S, M) with triangulation T_H.



$$x_1 x_1' = y_1(x_7 x_9) + x_2(x_8)$$

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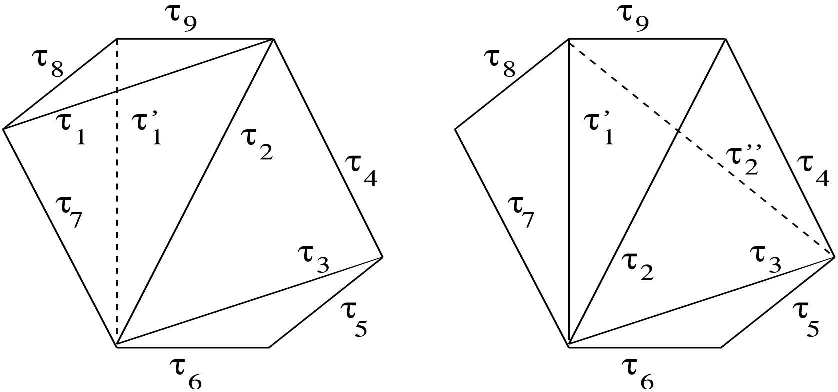

$$\begin{aligned} x_1 x_1' &= {\color{red}y_1}(x_7 x_9) + x_2(x_8) \\ x_2 x_2'' &= {\color{red}y_1 y_2} x_3(x_9) + x_1'(x_4) \end{aligned}$$

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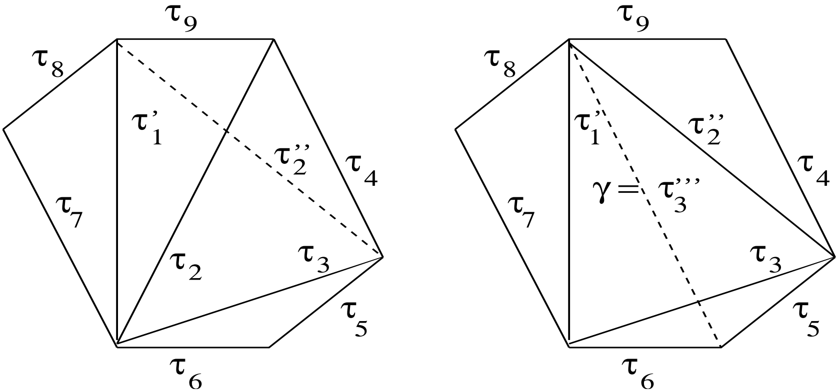
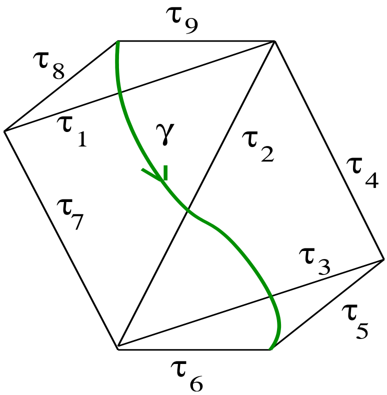


$$x_1 x_1' = y_1(x_7 x_9) + x_2(x_8)$$
$$x_2 x_2'' = y_1 y_2 x_3(x_9) + x_1'(x_4)$$
$$x_3 x_3''' = y_3 x_2''(x_6) + x_1'(x_5)$$

Example of Hexagon (continued)

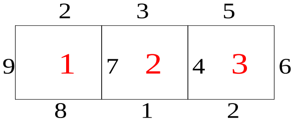


By using the Ptolemy relations on τ_1, τ_2, then τ_3, we obtain

$$x_3''' = x_\gamma = \frac{1}{x_1x_2x_3}\Big(x_2^2(x_5x_8) + y_1x_2(x_5x_7x_9) + y_3x_2(x_4x_6x_8) + y_1y_3(x_4x_6x_7x_9) + y_1y_2y_3x_1x_3(x_6x_9)\Big).$$

Example of Hexagon (continued)

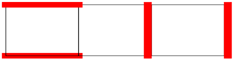
Consider the graph $G_{T_H,\gamma} =$



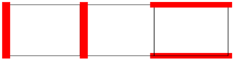
$G_{T_H,\gamma}$ has five perfect matchings $(x_4, x_5, \ldots, x_9 = 1)$:

$(x_9)x_1x_3(x_6)$,

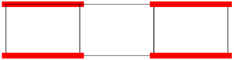
$(x_9x_7x_4x_6)$,



$x_2(x_8)(x_4x_6)$,



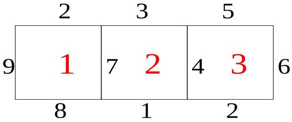
$(x_9x_7)x_2(x_5)$,



$x_2(x_8)x_2(x_5)$.

A perfect matching $M \subseteq E$ is a set of distinguished edges so that every vertex of V is covered exactly once. The weight of a matching M is the product of the weights of the constituent edges, i.e. $x(M) = \prod_{e \in M} x(e)$.

Example of Hexagon (continued)

Consider the graph $G_{T_H,\gamma} =$ 

$G_{T_H,\gamma}$ has five perfect matchings $(x_4, x_5, \ldots, x_9 = 1)$:

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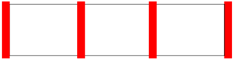
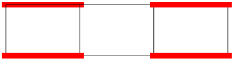
 $(x_9x_7x_4x_6)$,

 $x_2(x_8)(x_4x_6)$,

 $(x_9x_7)x_2(x_5)$,

 $x_2(x_8)x_2(x_5)$.

$$\frac{x_1x_3y_1y_2y_3 + y_1y_3 + x_2y_3 + x_2y_1 + x_2^2}{x_1x_2x_3}$$

These five monomials exactly match those appearing in the numerator of the expansion of x_γ. The denominator of $x_1x_2x_3$ corresponds to the labels of the three tiles.

A Graph Theoretic Approach

For every triangulation T (in a surface with or without punctures) and an ordinary arc γ through ordinary triangles, we construct a snake graph $G_{T,\gamma}$ such that

$$x_\gamma = \frac{\sum_{\text{perfect matching } M \text{ of } G_{T,\gamma}} x(M)y(M)}{x_1^{e_1(T,\gamma)} x_2^{e_2(T,\gamma)} \cdots x_n^{e_n(T,\gamma)}}.$$

x_γ is cluster variable (corresp. to γ w.r.t. seed given by T) with principal coefficients.

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$e_i(T,\gamma)$ is the crossing number of τ_i and γ (min. int. number),

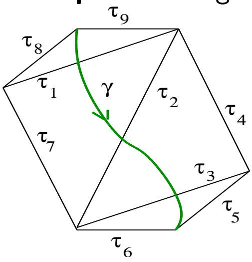
$x(M)$ is the weight of M,

$y(M)$ is the height of M (to be defined later),

Similar formula will hold for non-ordinary arcs (or through self-folded triangles).

Examples of $G_{T,\gamma}$

Example 1. Using the above construction for



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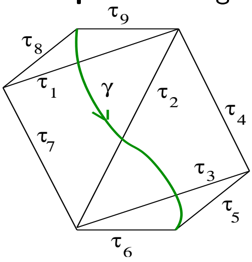

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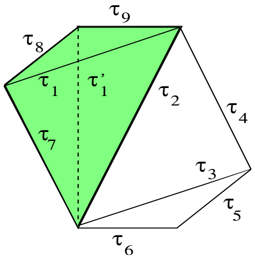

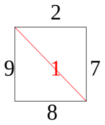

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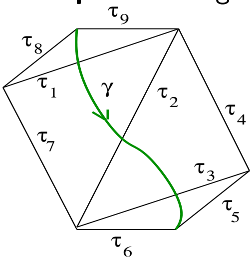

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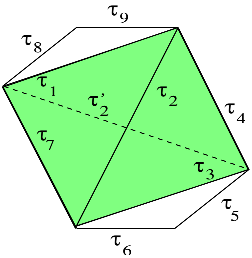

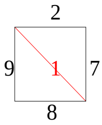

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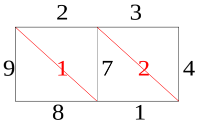

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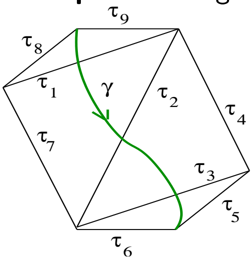

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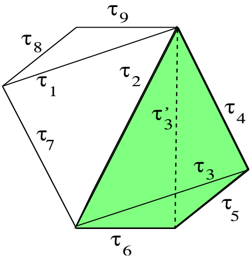

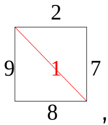

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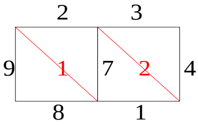

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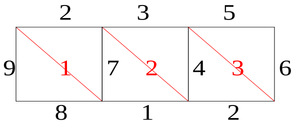
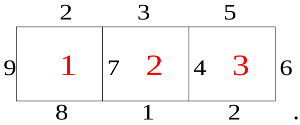


. Thus

$G_{T_H,\gamma} =$



. $x_\gamma|_{y_1=y_2=y_3=1} = \dfrac{x_1x_3 + 1 + 2x_2 + x_2^2}{x_1x_2x_3}$

Height Functions (of Perfect Matchings of *Snake* Graphs)

We now wish to give formula for $y(M)$'s, i.e. the terms in the F-polynomials.

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We use height functions which are due to William Thurston, and Conway-Lagarias.

Involves measuring contrast between a given perfect matching M and a fixed minimal matching M_-.

Height Function Examples

Recall that $G_{T_H,\gamma}$ has three faces, labeled 1, 2 and 3. $G_{T_H,\gamma}$ has five perfect matchings ($x_4, x_5, \ldots, x_9 = 1$):

$$x_\gamma = \frac{x_1 x_3 y_1 y_2 y_3 + y_1 y_3 + x_2 y_3 + x_2 y_1 + x_2^2}{x_1 x_2 x_3}$$

$y_1 y_2 y_3$,

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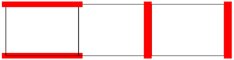
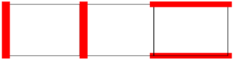


y_3,



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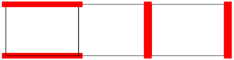
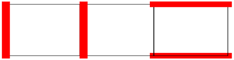
 y_3,  y_1,

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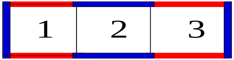

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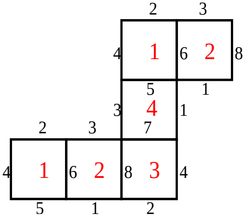
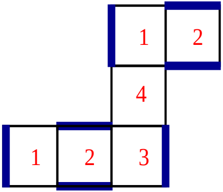
, and


Height Function Examples (continued)

For $G_{T_A,\gamma} =$ 

, M_{-} is 

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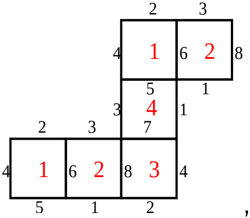
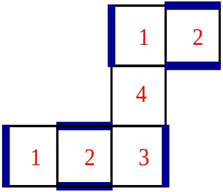
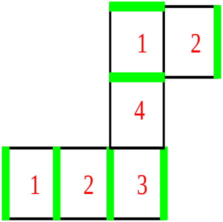
Height Function Examples (continued)

For $G_{T_{A,\gamma}} =$ 

, M_- is 

. One of the 17 matchings, M, is 

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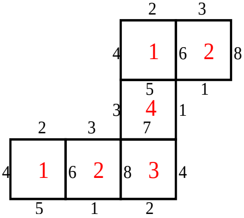
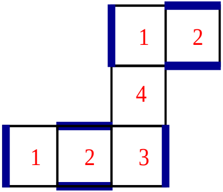
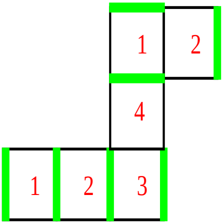
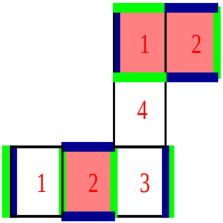
Height Function Examples (continued)

For $G_{T_A,\gamma} =$ 

, M_- is 

. One of the 17 matchings, M, is 

, so $M \ominus M_- =$ 

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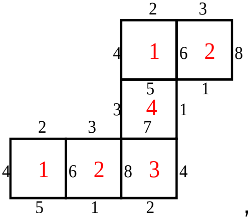
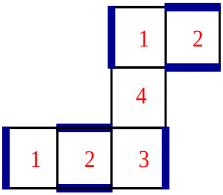
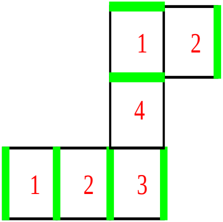
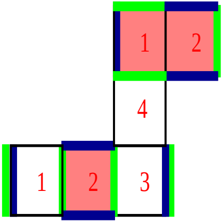
For $G_{T_A,\gamma} =$ 

, M_- is 

. One of the 17 matchings, M, is 

, so $M \ominus M_- =$ 

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which has height $y_1 y_2^2$. So one of the 17 terms in the cluster expansion of x_γ is (using FST convention) $\frac{x_4(x_6x_8)x_4(x_5)x_2(x_8)}{x_1^2x_2^2x_3x_4}(y_1y_2^2)$.

Summary

Theorem. (M-Schiffler-Williams 2009) For every triangulation T of a surface (with or without punctures) and an ordinary arc γ, we construct a snake graph $G_{\gamma,T}$ such that

$$x_\gamma = \frac{\sum_{\text{perfect matching } M \text{ of } G_{\gamma,T}} x(M)y(M)}{x_1^{e_1(T,\gamma)} x_2^{e_2(T,\gamma)} \cdots x_n^{e_n(T,\gamma)}}.$$

Here $e_i(T,\gamma)$ is the crossing number of τ_i and γ, $x(M)$ is the edge-weight of perfect matching M, and $y(M)$ is the height of perfect matching M. (x_γ is cluster variable with principal coefficients.)

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Corollary. The F-polynomial equals $\sum_M y(M)$, is positive, and has constant term 1.

The g-vector satisfies $\mathbf{x}^g = x(M_-)$.

Example 2 (Ordinary Arc through Self-folded Triangle)

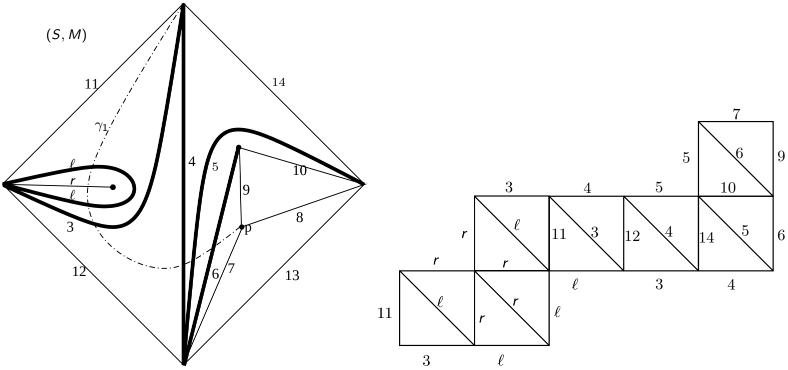

Figure: Ideal Triangulation T° of (S, M) and corresponding Snake Graph $\overline{G}_{T^\circ,\gamma_1}$.

Note the three consecutive tiles of our snake graph with labels ℓ, r and ℓ, as γ_1 traverses the loop ℓ twice and the enclosed radius r.

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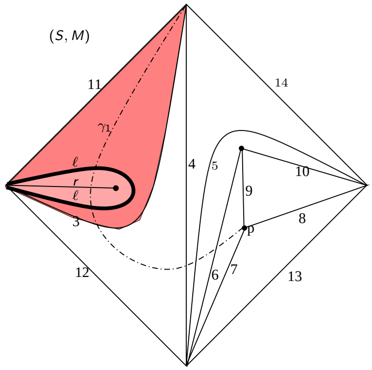

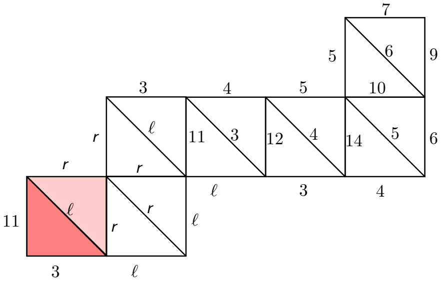

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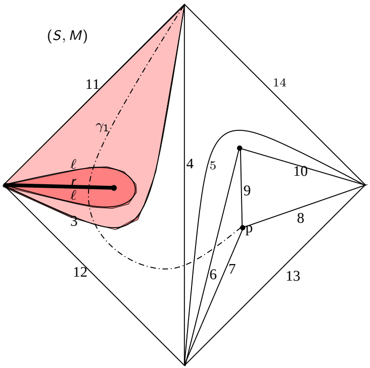

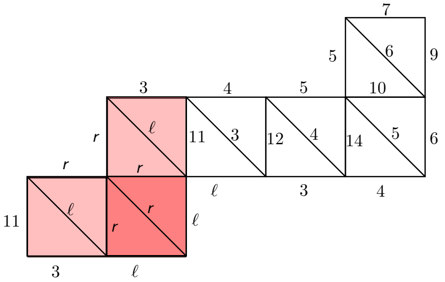

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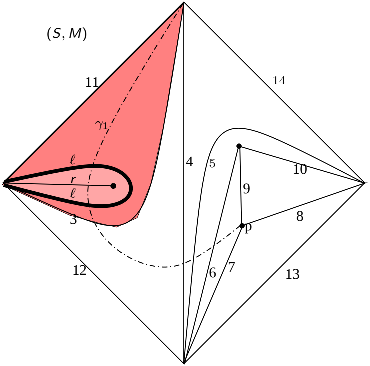

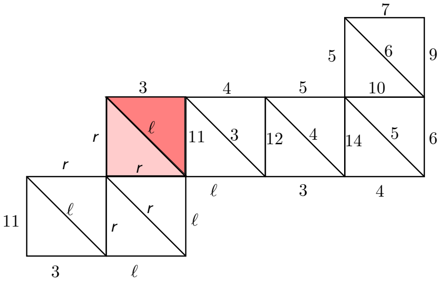

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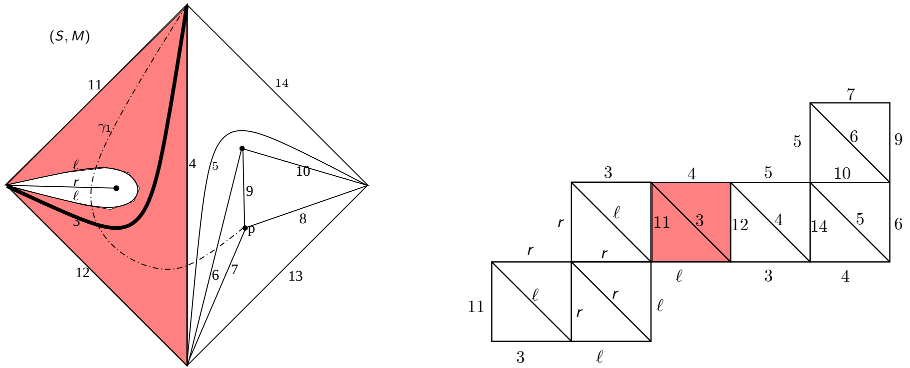

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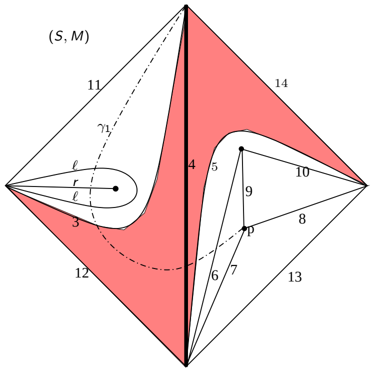

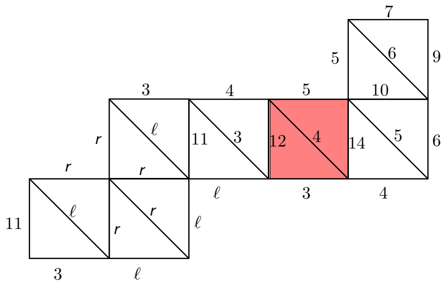

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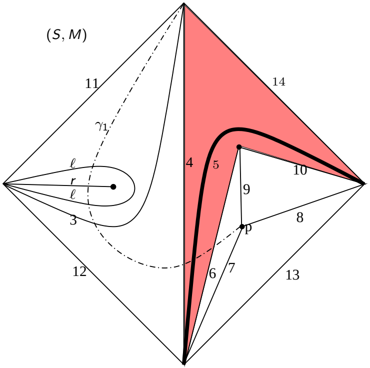

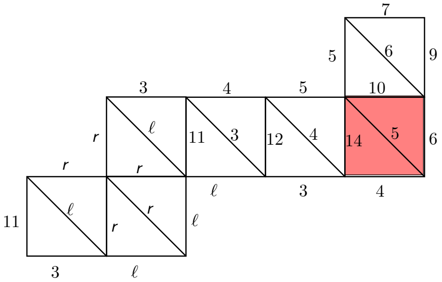

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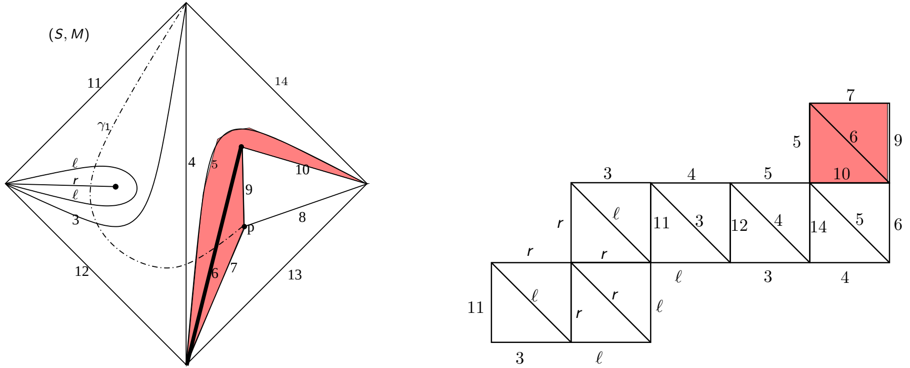


Example 3 (Notched Arc in Punctured Surface)

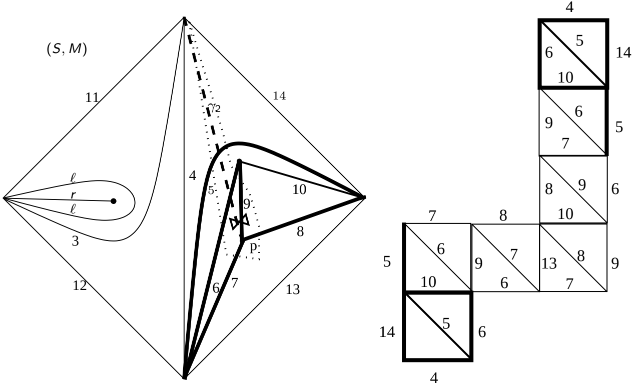


Figure: Ideal Triangulation T° of (S, M) and corresponding Snake Graph $\overline{G}_{T^\circ,\gamma_2}$.

We obtain the Laurent expansion for x_{γ_2} by summing over so called γ-symmetric matchings of G_{T°,γ_2}, those that agree on the two bold ends.

Thank You For Listening

Positivity for Cluster Algebras from Surfaces (with Ralf Schiffler and Lauren Williams), `arXiv:math.CO/0906.0748`

Cluster Expansion Formulas and Perfect Matchings (with Ralf Schiffler), `http://www-math.mit.edu/~musiker/PM.pdf`
(To appear in the Journal of Algebraic Combinatorics)

A Graph Theoretic Expansion Formula for Cluster Algebras of Classical Type, `http://www-math.mit.edu/~musiker/Finite.pdf`
(To appear in the Annals of Combinatorics)

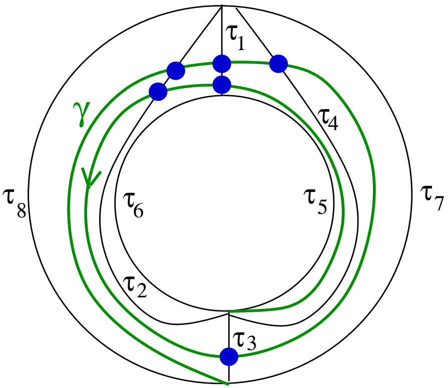
Fomin, Shapiro, and Thurston. *Cluster Algebras and Triangulated Surfaces I: Cluster Complexes*, Acta Math. 201 (2008), no. 1, 83–146.

Fomin and Zelevinsky. *Cluster Algebras IV: Coefficients*, Compos. Math. 143 (2007), no. 1, 112–164.

Slides Available at `http//math.mit.edu/~musiker/ClusterSurface.pdf`

Encore Example of $G_{T,\gamma}$: Annulus

Example 4. We now construct graph $G_{T_A,\gamma}$.



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