Homework 1 due on Monday October 12, 98

1. Let (Ω, F) be a measurable set and $\xi(\omega)$ be a function on Ω with values in a set X. Prove that

$$\{B \subset X : \xi^{-1}(B) \in F\}$$

is a σ-field.

2. Show that a subset K of a Polish space is a compact set if and only if it is closed and totally bounded.

3. Let $M = \{\mu_1, \mu_2, ...\}$ be a family of finite measures on a Polish space $(X, B(X))$ and let μ be a finite measure on $(X, B(X))$. Prove that if any sequence of elements of M has a subsequence weakly convergent to μ, then $\mu_n \overset{w}{\to} \mu$.

4. Let α be an irrational number. For $x \in R$, denote by $\{x\}$ the fractional part of x. Prove that for any $0 \leq a \leq b \leq 1$

$$\lim_{n\to\infty} \frac{1}{n} \#\{i = 1, 2, ..., n : \{i\alpha\} \in (a, b)\} = b - a.$$

5. Prove that the family of all finite dimensional cylindrical sets is an algebra. (Hint: attached points $t_1, ..., t_n$ and n may vary.)

6. Let Σ denote the cylindrical σ-field in the set of all X-valued functions on $[0, 1]$. Prove that for any $A \in \Sigma$ there exists a countable set $t_1, t_2, ... \in [0, 1]$ such that if $x. \in A$ and $y.$ is a function such that $y_{t_n} = x_{t_n}$ for all n, then $y. \in A$. In other words, elements of Σ are defined by specifying conditions on trajectories only at countably many points of $[0, 1]$.

7. Give an example of a Polish space $(X, B(X))$ such that the set $C([0, 1])$ of all bounded and continuous X-valued functions on $[0, 1]$ is not an element of the σ-field Σ from previous exercise. Thus you will see that there exists a very important and natural set which is not measurable.

For Homework 2 go to the next page.

Homework 2 due on Tuesday Nov 10, 98.

1. Let w_t be a one-dimensional Wiener process. Find

$$P\{\max_{s\leq 1} w_s \geq b, w_1 \leq a\}.$$

(Hint: the cases $a \leq b$ and $a > b$ are different.)

2. Let w_t be a one-dimensional Wiener process on a probability space $(\Omega, \mathcal{F}, P)$. Prove that

$$Ee^{w_t - t/2} = 1.$$

Introduce a new measure by $Q(d\omega) = e^{w_1 - 1/2} P(d\omega)$. Prove that $(\Omega, \mathcal{F}, Q)$ is a probability space, and that $w_t - t$ is a Wiener process on $(\Omega, \mathcal{F}, Q)$ on $[0, 1]$.

3. By using the results in Problem 2 show that

$$P\{\max_{s\leq 1}[w_s + s] \leq a\} = Ee^{w_1 - 1/2} I_{\max_{s\leq 1} w_s \leq a},$$

and by using the result in Problem 1, compute the last expectation.

4. Take π_t from Example 7.8 in lecture notes. Prove that for any $f \in L_2(0, 1)$ the stochastic integral of f against $\pi_t - t$ equals usual integral that is

$$-\int_0^1 f(s)\, ds + \sum_{\sigma_n \leq 1} f(\sigma(n)).$$

For Homework 3 go to the next page.

Homework 3 due on Tuesday Nov 24, 98.

1. Prove that if $f \in L_2(0,1)$, then $\int_0^t f(s)\,dw_s$ is a Gaussian process with zero mean and covariance

$$R(s,t) = \int_0^{s\wedge t} f^2(u)\,du = \left(\int_0^s f^2(u)\,du\right) \wedge \left(\int_0^t f^2(u)\,du\right).$$

2. Prove that the correlation function of second-order stationary process ξ_t is continuous if and only if the function ξ_t is continuous in t in the mean-square sense, that is as a function from (T,∞) to $L_2(\mathcal{F},P)$.

3. Remember the way the one-dimensional Riemann integral of continuous functions is defined. It turns out that this definition is easily extendible to continuous functions with values in Banach spaces. We mean the following definition.

Let $f(t)$ be a continuous function defined on a finite interval $[0,1]$ with values in a Banach space H. Then

$$\int_0^1 f(t)\,dt := \lim_{n\to\infty} \sum_{i=0}^{2^n-1} f(i2^{-n})2^{-n},$$

where the limit is understood in the sense of convergence in H. Assume that the limit exists indeed (the proof of this only uses the fact that continuous H-valued functions are uniformly continuous).

Similarly one defines the integrals over finite intervals $[a,b]$. The second-order stationary processes, we concentrate on, are assumed to be continuous as $L_2(\mathcal{F},P)$-valued functions. Therefore, for finite a and b, the integral $\int_a^b \xi_t\,dt$ is well-defined as the integral of an $L_2(\mathcal{F},P)$-valued continuous function ξ_t. We say that this is a mean-square integral.

By using the same method as in the proof of Theorem 9.9 in lecture notes, prove that if ξ_t is a second-order stationary process defined for all t, then

$$\underset{T\to\infty}{\text{l.i.m.}} \frac{1}{2T}\int_{-T}^{T} \xi_t\,dt$$

always exists. Also prove that this limit equals zero if and only if $F\{0\} = 0$. Finally prove that $F\{0\} = 0$ if $R(t) \to 0$ as $t \to \infty$. (Hint: For $R(0) = 1$ use $R(t) = Ee^{it\xi} = F\{0\} + Ee^{it\xi}I_{\xi\neq 0}$, and

$$\frac{1}{T}\int_0^T R(t)\,dt = F\{0\} + EI_{\xi\neq 0}[e^{iT\xi} - 1]/(iT\xi). \quad)$$

4. Let the spectral density $f(x)$ of a real-valued second-order stationary process ξ_t be rational, namely $f(x) = P_n(x)/P_m(x)$, where P_n and P_m are nonnegative polynomials of degree n and m respectively without common roots. Remember that we decomposed f as $\varphi(x)\bar{\varphi}(x)$ with $\varphi = P_+/Q_+$ and roots of P_+ and Q_+ lying in the closed upper half plane. Let $i\alpha_j$s be all roots of Q_+ and n_j be their multiplicities. Prove that for any $g \in L_2(\mathfrak{B}(\mathbb{R}),\ell)$,

$$\int_{\mathbb{R}} e^{itx} g(x)\varphi(x)\,dx = 0 \quad \forall t < 0 \Longrightarrow \int_{\mathbb{R}} g(x)\frac{1}{(ix+\alpha_j)^k}\,dx = 0 \quad k = 1,...,n_j.$$

(Hint: Define

$$G(t) = \int_{\mathbb{R}} e^{itx} g(x) \frac{1}{Q_+(x)}\, dx$$

and prove that G is $m/2-1$ times continuously differentiable in t and tends to zero as $|t| \to \infty$. Then prove that G satisfies the equation $P_+(-iD_t)G(t) = 0$ for $t \leq 0$, where $D_t = d/(dt)$. Solutions of this linear equations are linear combinations of some integral powers of t times exponential functions. Owing to the choice of P_+ its roots lie in the closed upper half plane which implies that the exponential function are of type $\exp(at)$ with $\operatorname{Re} a \leq 0$, none of which goes to zero as $t \to -\infty$. Since $G(t) \to 0$ as $t \to -\infty$, we get that $G(t) = 0$ for $t \leq 0$. Now apply linear diffrential operators to G to get the conclusion.)

For your final homework go to the next page.

Homework 4 = Take home final due on Tuesday Dec 8, 98.

1. Let Ω be a circle of length 1 centered at zero with Borel σ-field and linear Lebesgue measure. Fix a point $x_0 \in \Omega$ and think of any other point $x \in \Omega$ as the angle from x_0 to x in the clockwise direction. Then the operation $x_1 + x_2$ is well defined. Fix $\alpha \in \Omega$ and define $\xi_n(\omega) = \omega + n\alpha$. Since the distribution of $\omega + x$ is the same as that of ω for any x, we have that the distribution of $(\xi_0(\omega), ..., \xi_n(\omega))$ coincides with that of $(\xi_0(\omega + x), ..., \xi_n(\omega + x))$ for any x. Therefore, ξ_n is a stationary process. Prove that this process is ergodic. (Hint: For an invariant set A and any integer $m \in \mathbb{R}$ we have

$$\int_\Omega e^{2\pi i m\omega} I_A(\omega)\, d\omega = e^{2\pi i m\alpha} \int_\Omega e^{2\pi i m\omega} I_A(\omega)\, d\omega = e^{2\pi i mk\alpha} \int_\Omega e^{2\pi i m\omega} I_A(\omega)\, d\omega,$$

where $d\omega$ is the differential of the linear Lebesgue measure and k is any integer. By using the fact that for any integer $m \neq 0$

$$\frac{1}{n} \sum_{k=0}^{n} e^{2\pi i mk\alpha)} = \frac{e^{2\pi i m(n+1)\alpha} - 1}{n(e^{2\pi i m\alpha} - 1)} \to 0,$$

conclude that, for any square-integrable random variable f, $EfI_A = P(A)Ef$. Then take $f = I_A$.)

2. Prove the following.

Theorem 1. *If the process ξ_t is stochastically continuous on $[0, T]$ ($T < \infty$), then*

(i) it is uniformly stochastically continuous on $[0, T]$, that is for any $\gamma, \varepsilon > 0$ there exists $\delta > 0$ such that

$$P(|\xi_{t_1} - \xi_{t_2}| > \varepsilon) < \gamma,$$

whenever $t_1, t_2 \in [0, T]$ and $|t_1 - t_2| \leq \delta$;

(ii) it is bounded in probability on $[0, T]$.

3. Prove that if $x^n_\cdot \in D[0, \infty)$, $n = 1, 2, ...$, and x^n_t converge to x_t as $n \to \infty$ uniformly on each finite time interval, then $x_\cdot \in D[0, \infty)$. (Hint: Assume the contrary.)