# SYMMETRY GROUPS AND PATH-INDEPENDENT INTEGRALS

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ABSTRACT

Noether's general theorem gives a one-to-one correspondence between nontrivial conservation laws or path independent integrals for the Euler-Lagrange equations of some variational problem and the generalized variational symmetries of the variational problem itself, provided it satisfy certain nondegeneracy assumptions. Here we give a brief introduction to the theory of generalized symmetries and their connections with conservation laws. Applications are given to the classification of conservation laws for the equations of two dimensional elasticity, especially the linear isotropic and anisotropic cases.

## 1. INTRODUCTION

One of Professor Eshelby's lasting contributions to the study of dislocations and fracture mechanics was his discovery in 1956 of the celebrated energy-momentum tensor. It was the first example in a collection of four (seven in three dimensions) well-known and important path independent integrals that arise in both finite and linear elasticity, the applications of which are well documented in the other contributions to this memorial volume.

Subsequently, Günther (6) and Knowles and Sternberg (9) firmly established the group theoretic origins of these integrals by showing how they arise from the invariance of the underlying variational problem, under groups of translations, rotations and scaling symmetries, through a straight-forward application of Noether's Theorem relating symmetry groups to conservation laws. Although Knowles and Sternberg made claims that these are the only path independent integrals arising in this fashion, a closer analysis of their work shows that they employed only a limited version of the full power of Noether's Theorem. Indeed, in her widely quoted, but less widely appreciated paper (12), Emmy Noether not only gave the means to construct conservation laws from ordinary geometrical symmetry groups of the type studied by Günther and Knowles and Sternberg, she introduced the important concept of generalized symmetry groups, whose transformations depend on the deformation gradients and possibly higher order derivatives of the relevant dependent variables, AND showed that

ALL path independent integrals could be constructed from a knowledge of the generalized symmetries of the variational problem. Moreover, the fundamental infinitesimal methods introduced by Sophus Lie in his study of symmetry groups of differential equations, (10) - see also (13,18) - provide a systematic computational method of finding all such symmetry groups, and hence all path independent integrals. (A number of symbol-manipulating computer programs are now being developed to compute these symmetries - see (19,20) for instance - although as yet I am unaware of their extension to computing the corresponding integrals.)

To the best of my knowledge, despite the fact that Noether's Theorem has been available for well over 60 years, there was no attempt to apply this powerful result to any of the equations of elasticity until my own complete classification of the first order symmetries and conservation laws for the equations of two and three dimensional elasticity (14,15). The results are surprising. In three dimensions there are, in addition to the seven well-known conservation laws, six additional laws arising from generalized symmetries, except in a special case (when the Lamé moduli satisfy $7\mu + 3\lambda = 0$ ) where 13 additional laws result from generalized symmetries and ordinary conformal symmetries. (For $7\mu + 3\lambda \neq 0$ , these further laws still give rise to interesting divergence identities.) In two dimensions, there are whole families of conservation laws depending on a pair of arbitrary analytic functions. (These latter results were also indicated in some recent work of Tsamasphyros and Theocaris (21).)

The present paper consists of two parts. First we will review the general theory of generalized symmetries of differential equations, and Noether's general theorem relating these to conservation laws. One qustion that is of importance in the classification of conservation laws or symmetries is the question of triviality. Usually one is only interested in nontrivial conservation laws, but the issue of their precise relationship to nontrivial symmetries has not been dealt with adequately in the literature to date. Here we announce the result that for systems satisfying certain nondegeneracy assumptions there is a one-to-one correspondence between nontrivial generalized symmetries of a variational problem, and nontrivial conservation laws of the corresponding Euler-Lagrange equations. (Interestingly, according to some very recent results (17) this theorem is intimately related to the question of when a system of differential equations can be put into Cauchy-Kowalewski form.)

The second part of this paper deals with the applications of the general form of Noether's Theorem to the classification of path independent integrals for the equations of two dimensional elasticity. A complete analysis has only been completed for the linear case, but it is shown that for both isotropic and anisotropic linear elasticity, there are families of path independent integrals depending on pairs of arbitrary analytic functions. The precise structure of these integrals, though, does depend on whether or not the material is "equivalent" to an isotropic material or not. This latter analysis is based on a partial solution to the equivalence problem for linear, two dimensional elasticity: when are two problems equivalent under a linear change of variables in both the independent and dependent variables? The solution, perhaps surprising, indicates that for strongly elliptic problems there are, under this general notion of equivalence, only two independent invariants among the 16 elasticities. The case of nonlinear, two-dimensional elasticity has yet to be fully analyzed. However, striking similarities between the symmetry equations and the equations for conformal symmetries for Riemannian metrics leads to the conjecture that even in this case there will again be whole families of path independent integrals depending on two arbitrary analytic functions.

Of course, while this problem of classification of path independent integrals has some intrinsic interest, the real question is whether these new integrals have genuine applications to problems in fracture mechanics, dislocation theory, scattering of waves in elastic media and so on. Unfortunately, lack of time has precluded my addressing this important question in these papers, but it is an area that well deserves a concerted investigation, the results of which I hope to report on at a later date.

My thanks go to John Ball, who originally sparked my interest in applying Noether's Theorem to the problems of elasticity; and Professors Bilby, Miller and Rice for inviting me to participate in this conference.

## 2. SYMMERY GROUPS OF DIFFERENTIAL EQUATIONS

Let $x=(x^1,\ldots,x^p)$ , $u=(u^1,\ldots,u^q)$ be the independent and dependent variables in a system of differential equations

$$\Delta_i(x,u^{(n)})=0 \quad , \quad i=1,\ldots,\ell \ , \tag{1}$$

where $u^{(n)}$ denotes the partial derivatives $u^i_J = \partial^k u^i/\partial x^{j_1}\ldots\partial x^{j_k}$ of orders $0\leq k\leq n$ . A GEOMETRICAL SYMMETRY GROUP of the system is a connected (local) group of transformations $g:(x,u)\mapsto(\tilde{x},\tilde{u})$ with the property that if $u=f(x)$ is a solution, and $\tilde{u}=\tilde{f}(\tilde{x})$ is the function obtained from $f$ by transforming its graph by the group element $g$ , then $\tilde{f}=g\cdot f$ is also a solution. For instance, if $G$ is the group of rotations $g_\epsilon:(x,u)\mapsto(x\cos\epsilon+u\sin\epsilon \ , \ -x\sin\epsilon+u\cos\epsilon)$ , then

$\tilde{f} = g_\epsilon f$ is obtained from $f$ by rotating the graph of $f$ through the angle $\epsilon$.

Each one-parameter subgroup of $G$ is characterized by its infinitesimal generator $\underline{v} = \Sigma\, \xi^i(x,u)\partial/\partial x^i + \Sigma\, \varphi_j(x,u)\partial/\partial u^j$, the group elements being recovered by solving the system of ordinary differential equations $dx^i/d\epsilon = \xi^i$, $du^j/d\epsilon = \varphi_j$, $\epsilon$ being the group parameter. (For the rotation group $\underline{v} = u\,\partial/\partial x - x\,\partial/\partial u$.) Since $G$ acts on functions, it also acts on their derivatives - this defines the prolonged group action $pr^{(n)}g$: $(x,u^{(n)}) \mapsto (\tilde{x},\tilde{u}^{(n)})$, where $\tilde{u}^{(n)}$ are the derivatives of $\tilde{f}(\tilde{x})$. Similarly, an infinitesimal generator $\underline{v}$ prolongs to the space of derivatives:

$$pr\,\underline{v} = \underline{v} + \Sigma\, \varphi_j^J\, \partial/\partial u_J^j$$

where

$$\varphi_j^J = D_J(\varphi_j - \sum_i \xi^i u_i^j) + \sum_i \xi^i u_{J,i}^j$$

in which $u_i^j = \partial u^j/\partial x^i$, $u_{J,i}^j = \partial u_J^i/\partial x^i$, and $D_J = D_{j_1} D_{j_2} \cdots D_{j_k}$ is the total derivative (treating $u$ as a function of $x$).

THEOREM. If the system (1) is nondegenerate (see below), then $G$ is a symmetry group if and only if

$$pr\,\underline{v}(\Delta_i) = 0\ ,\quad i = 1,\ldots,\ell\ , \qquad (2)$$

whenever $\Delta = 0$.

The infinitesimal condition (2) for invariance yields a large number of elementary differential equations for the coefficient functions $\xi^i$, $\varphi_j$ of $\underline{v}$. In practice, there can always be solved, and hence the most general symmetry group of the system can be systematically computed - see (10,13,18) for examples.

DEFINITION. A system of differential equations is NONDEGENERATE if it satisfies

a) MAXIMAL RANK. The Jacobian matrix of $\Delta$ with respect to all variables $x, u^{(n)}$ is of rank $\ell$ everywhere.

b) LOCAL SOLVABILITY If $x_o, u_o^{(n)}$ are any fixed values satisfying $\Delta(x_o, u_o^{(n)}) = 0$, then there exists a solution $u = f(x)$ of the system with $u_o^{(n)} = f^{(n)}(x_o)$.

A result of Nirenberg (11) shows that quasi-linear elliptic systems are nondegenerate on a dense subset of $\{(x, u^{(n)}) : \Delta(x, u^{(n)}) = 0\}$ which is enough for the preceding theorem to be valid. A second class of nondegenerate systems are those in Cauchy-Kowalewski form

$$\frac{\partial^n u^i}{\partial t^n} = K_i(y, t, \tilde{u}^{(n)}) \ , \quad i = 1, \dots, q \ ,$$

for $K_i$ analytic, (2) in which $(y,t) = (y_i, \dots, y_{n-1}, t)$ is obtained from $x$ by a change of variables, and $\tilde{u}^{(n)}$ denotes all derivatives of $u$ of orders $\leq n$ except $\partial^n u^i / \partial t^n$. Any strongly elliptic system can, by a suitable change of variables, be put into Cauchy-Kowalewski form.

If we allow the coefficients $\xi^i, \varphi_j$ of the infinitesimal generator $\underline{v}$ to depend on derivatives of $u$, we have a GENERALIZED SYMMETRY. It is not difficult to see that we can assume, without loss of generality, that the symmetry is in EVOLUTIONARY FORM $\underline{v}_K = \Sigma K_j(x, u^{(m)}) \partial / \partial u^j$, in which $K_j = \varphi_j - \Sigma \xi^i u_i^j$. The corresponding group transformations are obtained by solving the system of evolution equations

$$\partial u^j / \partial \epsilon = K_j(x, u^{(m)}) \ , \quad u(x,0) = f(x) \ , \qquad (3)$$

with $f_\epsilon(x) = g_\epsilon . f(x) = u(x,\epsilon)$ . Again, $\underline{v}_K$ generates a symmetry group of (1), meaning that whenever $f(x)$ is a solution, so is $f_\epsilon(x)$ for all $\epsilon$ , if and only if (2) is satisfied for all solutions of $\Delta = 0$ . (This requires that the prolonged systems $\Delta^{(n)} = 0$ obtained by differentiating: $D_J \Delta_i = 0$ , are all nondegenerate - this still holds for elliptic systems and analytic Cauchy-Kowalewski - type systems). A symmetry $\underline{v}_K$ is TRIVIAL if $K = 0$ on solutions of $\Delta = 0$ ; two symmetries are EQUIVALENT if their difference is trivial, and we are really interested in equivalence classes of nontrivial symmetries.

## 3. CONSERVATION LAWS AND PATH-INDEPENDENT INTEGRALS

Given a system of differential equations (1), a CONSERVATION LAW is a divergence expression

$$\text{Div } P = \sum_{i=1}^{p} D_i P_i = 0 \ , \ P = P(x, u^{(k)}) \ , \tag{4}$$

which vanishes on all solutions $u = f(x)$ of the system. Each conservation law in two dimensions $(p = 2 \ , \ (x^1, x^2) = (x,y)$ , provides a path independent integral, namely by Green's theorem

$$\oint_C P(x,u^{(k)})dy - Q(x,u^{(k)})dx = 0 \tag{5}$$

for all closed curves $C$ provided $u = f(x)$ is a solution to the system. (In three dimensions, we end up with a "surface independent" integral $\int P \cdot dS$ .)

These are two types of TRIVIAL CONSERVATION LAWS: I) If $P = 0$ itself for all solutions of the system, then $\text{Div } P = 0$ for all solutions too; II) If $\text{Div } P = 0$ for ALL functions $u = f(x)$ , then the law is also trivial. An example of a conservation law of this latter type is $D_x(u_y) + D_y(-u_x) = 0$ - see (16) for a complete classification.

Two laws $P$ and $\tilde{P}$ are EQUIVALENT if their difference $P - \tilde{P}$ is a sum of trivial laws of the two types. As with symmetries, we are interested in classifying equivalence classes of nontrivial conservation laws.

Under the assumption of nondegeneracy, (4) is equivalent to the existence of functions $K_i^J(x,u^{(m)})$ such that

$$\mathrm{Div}\ P = \Sigma\, K_i^J\, D_J \Delta_i\ .$$

A simple integration by parts shows that there is an equivalent conservation law $\tilde{P}$ in CHARACTERISTIC FORM.

$$\mathrm{Div}\ \tilde{P} = \Sigma\, K_i \cdot \Delta_i\ , \tag{6}$$

where $K_i = \Sigma(-D)_J K_i^J$ is the CHARACTERISTIC of $\tilde{P}$ (and hence $P$). A characteristic is TRIVIAL if $K_i = 0$ on all solutions of the system (1); it can be seen that in (5) the characteristic $K = (K_1, \ldots, K_\ell)$ is uniquely defined up to addition of a trivial characteristic, so we should really talk about equivalence classes of characteristics as well.

THEOREM. If (1) is equivalent to a system in Cauchy-Kowalewski form, then there is a one-to-one correspondence between (equivalence classes of) nontrivial conservation laws and (equivalence classes of) nontrivial characteristics.

In other words, a nontrivial characteristic uniquely determines a nontrivial conservation law and vice versa. The requirement that the system be equivalent to one in Cauchy-Kowalewski form turns out to be essential; if this is not the case then either the local solvability condition for some prolongation $\Delta^{(m)} = 0$ is NOT satisfied, or there is a nontrivial relation of the form

$$\Sigma\, \mathcal{D}_\nu \Delta_\nu = 0 \tag{7}$$

among the equations, the $\mathcal{D}_\nu$ being certain differential operators. If the $\Delta_\nu$ actually arise as the Euler-Lagrange equations of some variational problem, then (7) means that Noether's second theorem, (12), is applicable, and there are nontrivial symmetry groups depending on arbitrary functions which give rise to only trivial conservation laws. The proofs of these statements will appear in (17).

## 4. NOETHER'S THEOREM

We now suppose that our system of differential equations are the Euler-Lagrange equations

$$E_i(W) = \delta\mathcal{W}/\delta u^i = 0 \quad , \quad i=1,\dots,q \tag{8}$$

for some variational problem $\mathcal{W}_{\Omega_0}[u] = \int_{\Omega_0} W(x,u^{(n)})dx$ . If $\mathcal{W}$ satisfies the Legendre-Hadamard condition, the system is nondegenerate.

DEFINITION. A generalized vector field $\underline{v}_K$ is a VARIATIONAL SYMMETRY of $\mathcal{W}$ if for every $\Omega \subset \Omega_0 \subset \mathbb{R}^p$ and every solution $u_\epsilon(x)$ of (3),

$$\mathcal{W}_\Omega[u_\epsilon] = \mathcal{W}_\Omega[u_0] + \int_0^\epsilon \mathcal{B}_{\partial\Omega}[u_{\epsilon'}]d\epsilon' \tag{9}$$

where

$$\mathcal{B}_{\partial\Omega}[u] = \int_{\partial\Omega} B(x,u^{(k)})\cdot dS$$

depends only on the boundary behavior of $u$ on $\partial\Omega$ .

Thus, up to the addition of boundary terms, $\mathcal{W}$ is invariant under the group action of $\underline{v}_K$ . (Another way of stating definition 4 is that $\mathcal{W}[u]$ is a conservation law for the evolution equations (3), i.e. $D_\epsilon\mathcal{W} + \mathrm{Div}\, X = 0$ for some flux $X(x,u^{(k)})$.)

LEMMA. A vector field $\underline{v}_K$ is a variational symmetry of $\mathcal{W}$ if and only if

$$\mathrm{pr}\,\underline{v}_K(W) = \mathrm{Div}\, B \tag{10}$$

for some p-tuple $B = (B_1,\dots,B_p)$ .

This is the form of variational symmetry proposed by Bessel-Hagen (1); Noether (12) omitted B (and hence the corresponding boundary term in (9)), but could no longer just consider evolutionary vector fields. (In this case (10) has the extra term $-L\,\mathrm{Div}\,\xi$ .) A simple computation shows that

$$\mathrm{pr}\,\underline{v}_K(L) = \Sigma\, K_i E_i(W) + \mathrm{Div}\, A \ ,$$

where $A = (A_1,\dots,A_p)$ depends on $K$ and $W$ ; the explicit form of $A$ is not required. As a result we immediately have Noether's Theorem.

THEOREM. There is a one-to-one correspondence between (equivalence classes of) nontrivial conservation laws and (equivalence classes of) variational symmetries; namely $K$ is the characteristic of a conservation law if and only if $\underline{v}_K$ is a variational symmetry.

Every variational symmetry of $\mathcal{W}$ is a symmetry of the Euler-Lagrange equations $E(W) = 0$ in the sense of section 2, but the most common counter-examples being groups of scaling transformations $(x,u) \longmapsto (\lambda x, \lambda^\alpha u)$. One practical method of finding variational symmetries, thus, is to first compute all symmetries of the Euler-Lagrange equations using Theorem 1, and then check which of these satisfy the additional variational condition (10). (There are, however, more direct ways of doing this - see (13).)

## 5. TWO DIMENSIONAL ELASTICITY

For simplicity, we treat the path-independent integrals for two-dimensional homogeneous hyperelastic materials in the abscence of body forces. Thus the variational problem is

$$\mathcal{W} = \int_\Omega W(\nabla \underline{u})\,dxdy$$

in which $W$ is the stored energy function, $\underline{x} = (x,y)$ the material coordinates, $\underline{u} = (u,v)$ the deformation, so $\nabla\underline{u} = (u_x, u_y; v_x, v_y) \equiv (p,q;r,s)$ is the deformation gradient. The Euler-Lagrange equations are the second order system

$$E_u(W) = D_x W_p + D_y W_q = 0 ,$$

$$E_v(W) = D_x W_p + D_y W_s = 0 .$$

The goal is to analyze all first order path independent integrals, i.e. those of the form

$$\oint P\,dy - Q\,dx$$

in which $P$ and $Q$ are functions of $\underline{x}, \underline{u}, \nabla\underline{u}$.

For computational purposes, it is advantageous to begin by looking at those integrals in which $P$ and $Q$ depend solely on the deformation gradient $\nabla\underline{u}$ (see (15).) In this case, Noether's theorem says that $P(\nabla\underline{u})$, $Q(\nabla\underline{u})$ are the components of a conservation law if and only if there exists functions $K(\nabla\underline{u})$, $L(\nabla\underline{u})$ such that

$$\begin{aligned} P_p &= \alpha K + \beta L , \quad P_q + Q_p = \alpha' K + \beta' L , \quad Q_q = \alpha'' K + \beta'' L , \\ P_r &= \beta K + \gamma L , \quad P_s + Q_r = \beta' K + \gamma' L, \quad Q_s = \beta'' K + \gamma'' L , \end{aligned} \tag{11}$$

where

$$\alpha = W_{pp}\ ,\ \beta = W_{pr}\ ,\ \gamma = W_{rr}\ ,\ \alpha' = 2W_{pq}\ ,\ \beta' = W_{ps} + W_{qr}\ ,$$
$$\gamma' = 2W_{rs}\ ,\ \alpha'' = W_{qq}\ ,\ \beta'' = W_{qs}\ ,\ \gamma'' = W_{ss}\ .$$

These equations bear a remarkable similarity to a "vector version" of the equations for a conformal symmetry of a Riemannian metric, (3), and are so called the two-dimensional VECTOR CONFORMAL EQUATIONS. To date, no progress has been made to their solution for general nonlinear $W$ , although the following CONJECTURE seems plausible: For the two-dimensional vector conformal equations there are an infinite family of solutions, hence an infinite family of path independent integrals with $P,Q$ depending only on $\nabla \underline{u}$ . (In the analogous situation for two dimensional Riemannian metrics, this result is true, since every such metric is conformal to a flat metric, in which case, each complex analytic function provides a conformal symmetry.) Finally, if $P,Q$ are a solution of (1), then $\underline{v} = K\partial_u + L\partial_v$ is a (generalized) variational symmetry of $\mathcal{W}$ .

For quadratic $W(\nabla \underline{u})$ , leading to the equations of linear two dimensional elasticity, the situation is much better understood. Here the coefficients $\alpha,\dots,\gamma''$ in the vector conformal equations are related to the more usual elasticity constants $c_{ijk\ell}$ according to the rule

$$\alpha = c_{1111},\ \beta = c_{1121}\ ,\ \gamma = c_{2121},\ \alpha' = 2c_{1112},\ \beta' = c_{1122} + c_{1221}\ ,$$
$$\gamma' = 2c_{2122}\ ,\ \alpha'' = c_{1212}\ ,\ \beta'' = c_{1222}\ ,\ \gamma'' = c_{2222}\ .$$

Of course, the $c_{ijk\ell}$'s obey additional symmetry properties when arising from a theory of linear elasticity (7) leading to the following relations among $\alpha,\dots,\zeta$ :

$$2\beta = \alpha'\ ,\ 2\beta'' = \gamma'\ ,\ \gamma = \alpha''\ ,$$

but these do not appear to be especially relevant to the subsequent analysis.

Let $\nabla P = (P_p, P_q, P_r, P_s)^T$ , and similarly for $\nabla Q$ .

Eliminating $K$ and $L$ from (10), we obtain a system of the form

$$M \nabla P = N \nabla Q \qquad (12)$$

in which $M$ and $N$ are $4 \times 4$ matrices whose entries depend on the constants $\alpha,\dots,\zeta$ . (The precise expressions are easy to write down, but rather messy.) The nature of the solutions to (12) depends on the structure of the eigenvalues of the matrix $M^{-1}N$ ; in general it can be shown that $M^{-1}N$ has two pairs of complex conjugate eigenvalues $k_1 \pm i\ell_1$ and $k_2 \pm i\ell_2$. If these pairs are distinct, then $M^{-1}N$ is diagonalizable, otherwise we will (usually) have a nondiagonal Jordan canonical form. The structure of the space of path-independent integrals is different in each case.

THEOREM. Given the matrix $M^{-1}N$ there exist two independent complex-linear combinations $\xi$ and $\eta$ of $p,q,r,s$ such that the complex function $F = F(\nabla \underline{u}) = P + iQ$ is a conservation law if and only if

a) In case $M^{-1}N$ is diagonalizable, with distinct eigenvalues

$$F = F_1(\xi) + F_2(\eta)$$

b) In case $M^{-1}N$ is not diagonalizable

$$F = \xi F_1(\eta) + \overline{F_1(\eta)} + F_2(\eta) ,$$

where $F_1, F_2$ are analytic in $\eta$.

(The combinations $\xi, \eta$ can be constructed from the eigenvectors to the matrix $M^{-1}N$.)

It is thus important to know whether a given elastic material is in the diagonalizable or nondiagonalizable case. The only case analyzed to date, the case of linear isotropic materials is nondiagonalizable (15). To attempt any analysis of nonisotropic materials it is necessary to simplify the constants $c_{ijk\ell}$ as much as possible.

## 6. THE EQUIVALENCE PROBLEM

In its general form, the EQUIVALENCE PROBLEM is connected with the question of when two variational problems $\mathscr{V}$ and $\mathscr{V}'$ are the same under a change of both independent and dependent variable. (See section 6 of (5) for an "elementary" case.). Here we are first interested in the special two-dimensional quadratic equivalence problem: When are

$$\int W(\nabla \underline{u})\, d\underline{x} \quad \text{and} \quad \int \tilde{W}(\nabla \tilde{\underline{u}})\, d\tilde{\underline{x}} ,$$

for $W, \tilde{W}$ quadratic in the deformation gradient, equivalent under a linear change of variables:

$$\tilde{\underline{x}} = A\underline{x} , \quad \tilde{\underline{u}} = B\underline{u} \; ? \qquad (13)$$

Equivalently, when are two sets of elasticities $c_{ijk\ell}$ and $\tilde{c}_{ijk\ell}$

related under such a change of variables? Included in the equivalence problem is the problem of finding simple (or canonical) forms of variational problems: Given $\int W(\nabla \underline{u})d\underline{x}$ , find A,B such that (13) makes $\int \tilde{W}(\nabla \tilde{\underline{u}})d\tilde{\underline{x}}$ as simple as possible. Even the relatively easy case of two independent and two dependent variables presents a number of difficulties, and so far I have only incomplete, but nevertheless intriguing results.

DEFINITION. A two-dimensional problem is called QUASI-ISOTROPIC if its elasticities have the form

$$c_{1111} = c_{2222} = 2\mu + \lambda \ , \ c_{1212} = c_{2121} = c_{1221} = c_{2112} = \mu \ ,$$

$$c_{1122} = c_{2211} = \nu \ ,$$

for some constants $\mu, \lambda, \nu$ , with all unspecified elasticities vanishing.

In particular, if $\nu = \lambda$ , then the material is isotropic. Note that actually only two of the constants $\mu, \lambda, \nu$ are arbitrary, since by rescaling $\underline{u}$ we can always arrange that $\mu = 1$ , say. It is easy to check that a quasi-isotropic material is strongly elliptic if and only if

$$\mu > 0 \ , \ -4\mu - \lambda < \nu < 2\mu + \lambda \ .$$

THEOREM. Every quadratic $W(\nabla \underline{u})$ satisfying the Legendre-Hadamard condition and sufficiently close to the linear isotropic case is equivalent to a strongly elliptic quasi-isotropic problem.

The phrase "close to isotropic" means that the elasticities $c_{ijk\ell}$ do not differ too much from the isotropic elasticities. I conjecture that the theorem remains true if this condition is dropped i.e. any strongly elliptic quadratic $W(\nabla \underline{u})$ is equivalent to a quasi-isotropic one, but to prove this looks rather complicated. The present proof relies on some Lie-algebraic tools and a generalization of Frobenius' theorem due to Hermann (8). Unfortunately, the proof is nonconstructive; it gives no clues as to how to find the requisite matrices A,B such that (12) transforms $W(\nabla \underline{u})$ into a quasi-isotropic problem. The values of $\lambda, \nu$ , however, can be found using invariant theory.

Note that in principle the theorem says that there are only two independent elasticities in two-dimensional elasticity, up to the above generalized notion of equivalence. (Contrast this with the more standard six independent elasticities (7) when one just uses the basic symmetry relations on the subscripts $c_{ijk\ell}$ .)

Returning to our classification of path independent integrals it is not to difficult to prove that for a strongly elliptic quasi-isotropic problem, the eigenvalues of the matrix $M^{-1}N$ of section 5 are distinct unless $\nu = \lambda$ or $\nu = -2\mu - \lambda$. The first is the isotropic case, the second equivalent under a reflection $(u,v) \longmapsto (u,-v)$.

PROPOSITION. If $W(\nabla \underline{u})$ is strongly elliptic, then $M^{-1}N$ is diagonalizable unless $W(\nabla u)$ is equivalent to an isotropic material.

(Of course, this is subject to the establishment of our earlier conjecture.) Thus the isotropic materials are thus distinguished by the structure of their space of path independent integrals. (This is probably only true in two dimensional elasticity!)

## 7. FURTHER INTEGRALS.

So far we have concentrated on path-independent integrals in which $\oint P\,dy - Q\,dx$ is such that $P + iQ = F$ depends only on $\nabla \underline{u}$. If we now relax the requirements so that $F$ depends on $\underline{x}, \underline{u}$ and $\nabla \underline{u}$, then in (15) it was shown that for linear isotropic elasticity all such integrals were given as follows.

THEOREM For linear, isotropic two-dimensional elasticity, every path independent integral is given by a linear combination on the following

$$2\mu(2\mu+\lambda)\xi\partial F_1/\partial\eta + (\mu+\lambda)i\overline{F_1} + F_2 ,$$

$$i(\tilde{w}\eta - w\tilde{\eta}) , \quad (4\mu(2\mu+\lambda)w - (\mu+\lambda)iz\eta)\eta ,$$

where $z = x + iy$, $w = u + iv$, $\xi = (u_x - v_y) + i(u_y + v_x)$, $\eta = \mu(v_y - u_x) + i(2\mu+\lambda)(u_x + v_y)$; and $F_1(\bar{z},\eta)$, $F_2(\bar{z},\eta)$ are analytic in their arguments, and $\hat{w}(x,y)$ is an arbitrary solution of the system with corresponding $\tilde{\eta}(x,y)$.

A similar result holds for anisotropic materials, but I have not completed the details in it.

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