On the Structure and Generators of Differential Invariant Algebras

Peter J. Olver
University of Minnesota
http://www.math.umn.edu/∼olver

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Invariants

Definition. Let G be a group acting on a space M. Then an invariant is a real-valued function $I : M \to \mathbb{R}$ that does not change under the action of G:

$$I(g \cdot z) = I(z) \quad \text{for all} \quad g \in G, \quad z \in M$$

⋆ If G acts transitively, there are no (non-constant) invariants.

Differential Invariants

Let M be a smooth manifold. Given a smooth submanifold (curve, surface, ...)

$$S \subset M$$

a differential invariant is an invariant $I : \mathrm{J}^n \;\longrightarrow\; \mathbb{R}$ of the prolonged action of G on its derivatives (jets):

$$\boxed{I(g \cdot z^{(n)}) \;=\; I(z^{(n)})}$$

$\mathrm{J}^n = \mathrm{J}^n(M,p)$ — jet space for p-dimensional submanifolds;

Local coordinates: $z^{(n)} = (x, u^{(n)}) = (\ldots x^i \ldots u^\alpha_J \ldots)$ represent partial derivatives of the submanifold $S = \{\, u = f(x) \,\}$.

Applications of Differential Invariants

- Equivalence and signatures of submanifolds
  $\Longrightarrow$ image processing, jigsaw puzzles
- Moduli spaces
- Invariant differential equations
- Integration of ordinary differential equations
- Group splitting/foliation of PDEs
  — explicit solutions & Bäcklund transformations
- Conservation laws and characteristic classes
- Invariant variational problems
  $\Longrightarrow$ Noether's Two Theorems

Examples of Differential Invariants

Euclidean Group Acting on $\mathbb{R}^3$

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$$\Longrightarrow \quad \text{group of rigid motions}$$

$$z \quad \longmapsto \quad R\,z + b \qquad\qquad R \in \mathrm{SO}(3)$$

- Induced action on curves and surfaces.

Euclidean Space Curves $\quad C \subset \mathbb{R}^3$

- κ — curvature: order $= 2$
- τ — torsion: order $= 3$
- $\kappa_s, \tau_s, \kappa_{ss}, \ldots$ — derivatives w.r.t. arc length ds

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Theorem. Every Euclidean differential invariant of a space curve $C \subset \mathbb{R}^3$ can be written

$$I = H(\kappa, \tau, \kappa_s, \tau_s, \kappa_{ss}, \ \ldots\ )$$

Thus, κ and τ *generate* the differential invariants of space curves under the Euclidean group.

Euclidean Surfaces $\quad S \subset \mathbb{R}^3$

- κ_1, κ_2 — principal curvatures: order $= 2$
- $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ — mean curvature
- $K = \kappa_1 \kappa_2$ — Gauss curvature
- $\mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots$ — derivatives with respect to the equivariant Frenet frame on S

Theorem. Every Euclidean differential invariant of a non-umbilic surface $S \subset \mathbb{R}^3$ can be written

$$I = \Phi(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots\ )$$

Thus, H, K *generate* the differential invariant algebra of (generic) Euclidean surfaces.

Equi-affine Group on $\mathbb{R}^3$

$$G = \mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3 \qquad \text{— volume preserving}$$

$$z \longmapsto A\,z + b, \qquad \det A = 1$$

Curves in $\mathbb{R}^3$:

- κ — equi-affine curvature: order $= 4$
- τ — equi-affine torsion: order $= 5$
- $\kappa_s, \tau_s, \kappa_{ss}, \ldots$ — diff. w.r.t. equi-affine arc length

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Surfaces in $\mathbb{R}^3$:

- P — Pick invariant: order $= 3$
- $Q_0, Q_1, \ldots, Q_4$ — fourth order invariants
- $\mathcal{D}_1 P, \mathcal{D}_2 P, \mathcal{D}_1 Q_\nu, \ldots$ diff. w.r.t. the equi-affine frame

Basic Framework

G — Lie group (or Lie pseudo-group)
acting on a manifold M

$\mathcal{I}(G)$ — "algebra" of all differential invariants for
p-dimensional submanifolds $S \subset M$

Goal: Describe the structure of $\mathcal{I}(G)$ in as much
detail as possible.

The Basis Theorem

Theorem. The differential invariant algebra $\mathcal{I}(G)$ is locally generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim S$ invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa .$$

★ Lie groups: *Lie, Ovsiannikov, Fels–PJO*

★ Lie pseudo-groups: *Tresse, Kumpera, Pohjanpelto–PJO, Kruglikov–Lychagin*

General Problems

Determine the structure of the algebra of differential invariants: generators, syzygies, commutators, etc.

Find a minimal system of generating differential invariants.

Key Issues

- Basis of generating invariants: $I_1, \ldots, I_\ell$
- Commutation formulae for

  the invariant differential operators:

$$[\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y^i_{jk}\, \mathcal{D}_i$$

$\Longrightarrow$ Non-commutative differential algebra

- Syzygies (functional relations) among

  the differentiated invariants:

$$\Phi(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ ) \equiv 0$$

$\Longrightarrow$ Codazzi relations

Curves

Theorem. Let G be an ordinary* Lie group acting on the m-dimensional manifold M. Then, locally, there exist $m-1$ generating differential invariants $\kappa_1, \ldots, \kappa_{m-1}$. Every other differential invariant can be written as a function of the generating differential invariants and their derivatives with respect to the G-invariant arc length element ds.

* ordinary = transitive + no pseudo-stabilization.

Euclidean Surfaces

Theorem.

The algebra of Euclidean differential invariants for suitably non-degenerate surfaces is generated by only the mean curvature through invariant differentiation.

In particular:

$$K = \Phi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \ \ldots\ )$$

Equi-affine Surfaces

Theorem.

The algebra of equi-affine differential invariants for non-degenerate surfaces is generated by the Pick invariant through invariant differentiation.

In particular:

$$Q_\nu = \Phi_\nu(P, \mathcal{D}_1 P, \mathcal{D}_2 P, \ \ldots \ )$$

Further Results

For suitably non-degenerate surfaces $S \subset \mathbb{R}^3$:

Theorem. $G = \mathrm{SO}(4,2)$

The algebra of conformal differential invariants is generated by a single third order differential invariant.

Theorem. $G = \mathrm{PSL}(4)$

The algebra of projective differential invariants is generated by a single fourth order differential invariant.

$\Longrightarrow$ (*with Evelyne Hubert*)

Theorem. $G = \mathrm{GL}(3)$

The algebra of differential invariants for ternary forms is generated by a single third order differential invariant.

$\Longrightarrow$ (*with Gülden Gün Polat*)

Example. $G\colon\ (x,y,u) \longmapsto (\,x+a, y+b, u+P(x,y)\,)$

$a,b\in\mathbb{R}$, $\quad P$ is an arbitrary polynomial of degree $\leq n$

Differential invariants:

$$u_{i,j} = \frac{\partial^{i+j}u}{\partial x^i \partial y^j} \qquad i+j \geq n+1$$

Invariant differential operators:

$$\mathcal{D}_1 = D_x \qquad \mathcal{D}_2 = D_y$$

Minimal generating set:

$$u_{i,j} \qquad i+j = n+1$$

♠ For submanifolds of dimension $p \geq 2$, the number of generating differential invariants can be arbitrarily large.

For surfaces and higher dimensional submanifolds, there is as yet no criterion for determining whether a given generating set of differential invariants is minimal!

- Except when there is a single generator.

$\Longrightarrow$ Moving frames furnish constructive algorithms for determining the full structure of the differential invariant algebra $\mathcal{I}(G)$ and hence solve the preceding problems!

Equivariant Moving Frames

Definition. An $n^{\rm th}$ order moving frame is a (locally) G-equivariant map

$$\rho^{(n)} : V^n \subset \mathrm{J}^n \longrightarrow G$$

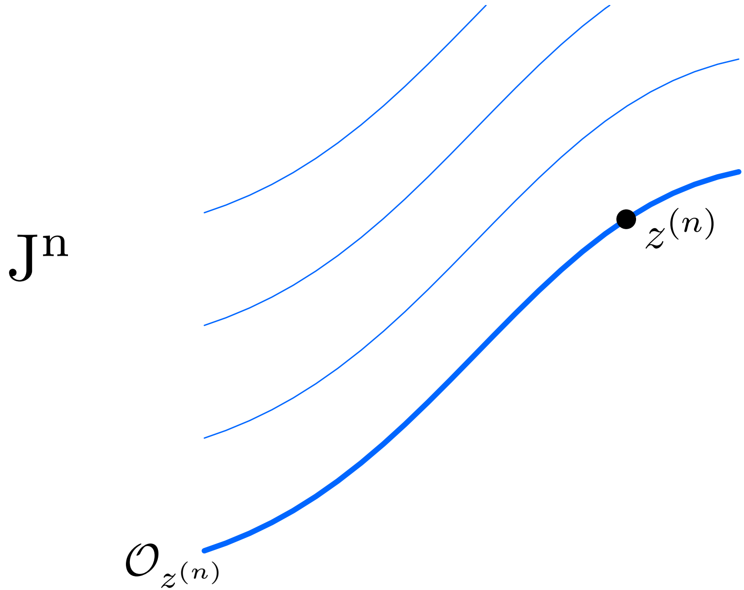
Right equivariance:

$$\rho(g^{(n)} \cdot z^{(n)}) = \rho(z^{(n)}) \cdot g^{-1}$$

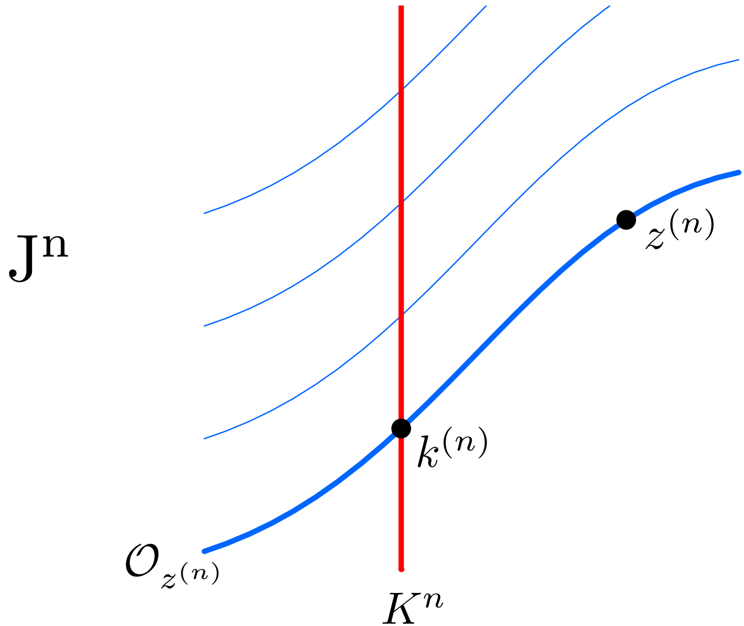
- *Élie Cartan*
- *Guggenheimer, Griffiths, Green, Jensen*
- *Fels, Kogan, Pohjanpelto, PJO*

Theorem. Given $n \gg 0$ sufficiently large, there is a dense open subset $U^N \subset \mathrm{J}^n$ such that a moving frame exists in a neighborhood of any jet $z^{(n)} \in U^n$.

Geometric Construction

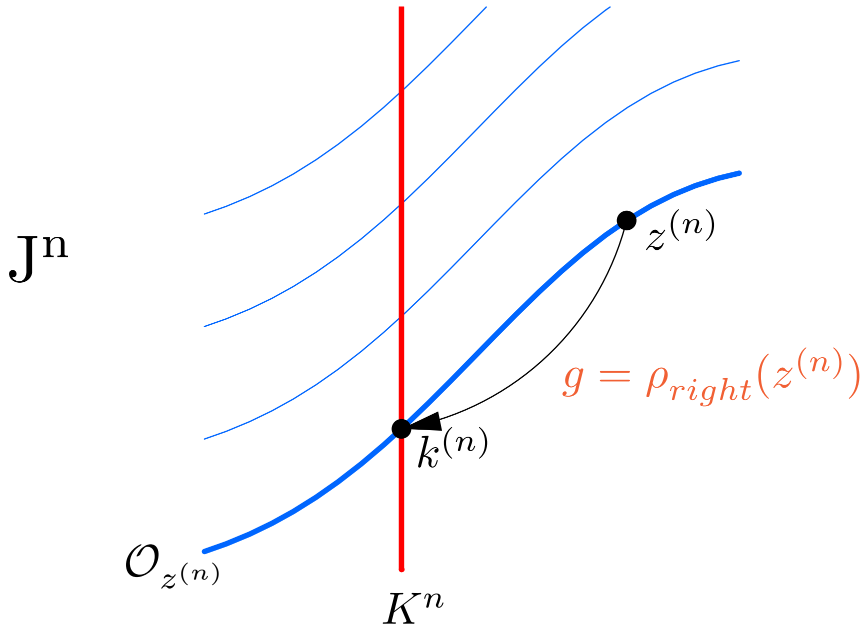


Geometric Construction



Normalization = choice of cross-section to the group orbits

Geometric Construction



Requires that G acts locally freely st $z^{(n)}$

Algebraic Construction

1. Write out the explicit formulas for the prolonged group action:

$$w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}$$

$\Longrightarrow$ *Implicit differentiation*

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2. From the components of $w^{(n)}$, choose $r = \dim G$ *normalization equations* to define the cross-section:

$$w_1(g, z^{(n)}) = c_1 \qquad \ldots \qquad w_r(g, z^{(n)}) = c_r$$

3. Solve the normalization equations for the group parameters $g = (g_1, \ldots, g_r)$:

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

The solution is the right moving frame.

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4. Invariantization: Substitute the moving frame formulas

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

for the group parameters into the un-normalized components of $w^{(n)}$ to produce a complete system of functionally independent differential invariants of order $\leq n$:

$$I_k(x, u^{(n)}) = w_k(\rho(z^{(n)}), z^{(n)})), \qquad k = r + 1, \ \ldots \ , \dim \mathrm{J}^n$$

The Fundamental Differential Invariants

Invariantized jet coordinate functions:

$$\boxed{H^i(x,u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x,u^{(l)}) = \iota(u^\alpha_K)}$$

- The constant differential invariants $Z^\sigma = c^\sigma$, $\sigma = 1,\ldots,r$, as defined by the chosen cross-section normalizations, are known as the phantom invariants.
- The remaining non-constant differential invariants are the basic invariants and form a complete system of functionally independent differential invariants for the prolonged group action.

Invariantization of general differential functions:

$$\iota\,[\,F(\,\ldots\;x^i\;\ldots\;u_J^\alpha\;\ldots\,)\,] = F(\,\ldots\;H^i\;\ldots\;I_J^\alpha\;\ldots\,)$$

The Replacement Theorem: (Rewrite Rule)

If J is a differential invariant, then $\iota(J) = J$.

$$J(\,\ldots\;x^i\;\ldots\;u_J^\alpha\;\ldots\,) = J(\,\ldots\;H^i\;\ldots\;I_J^\alpha\;\ldots\,)$$

Key fact: Invariantization and differentiation do not commute:

$$\iota(D_i F) \;\neq\; \mathcal{D}_i \iota(F)$$

Recurrence Formulae

$$\mathcal{D}_j\, \iota(F) = \iota(D_j F) \; + \; \sum_{\kappa=1}^{r} R_j^\kappa \, \iota(\mathbf{v}_\kappa(F))$$

$F = F(x, u^{(n)})$ — differential function

$\iota(F)$ — invariantization (differential invariant)

$\mathcal{D}_i = \iota(D_{x^i})$ — invariant differential operators

$\mathbf{v}_1, \ldots, \mathbf{v}_r$ — prolonged infinitesimal generators

R_j^κ — Maurer–Cartan invariants

Recurrence Formulae

$$\mathcal{D}_j\, \iota(F) = \iota(D_j F) \; + \; \sum_{\kappa=1}^{r} R_j^\kappa\, \iota(\mathbf{v}_\kappa(F))$$

♠ If $\iota(F) = c$ is a phantom differential invariant, then the left hand side of the recurrence formula is zero. The collection of all such phantom recurrence formulae form a linear algebraic system of equations that can be uniquely solved for the Maurer–Cartan invariants R_j^κ!

♡ Once the Maurer–Cartan invariants are replaced by their explicit formulae, the induced recurrence relations completely determine the structure of the differential invariant algebra $\mathcal{I}(G)$!

Commutator Invariants

$$\left[\,\mathcal{D}_j, \mathcal{D}_k\,\right] = \sum_{i=1}^{p} Y^i_{jk}\, \mathcal{D}_i$$

- As a consequence of the recurrence formulae for the invariant coframe $\omega^i = \iota(dx^i)$:

$$Y^i_{jk} = -\,Y^i_{kj} = \sum_{\kappa=1}^{r} \left[\, R^\kappa_k\, \iota(D_j \mathbf{v}_\kappa(x^i)) - R^\kappa_j\, \iota(D_k \mathbf{v}_\kappa(x^i))\,\right]$$

The Symbolic Moving Frame Calculus

Thus, remarkably, the structure of $\mathcal{I}(G)$ can be determined without knowing the explicit formulae for either the moving frame, or the differential invariants, or the invariant differential operators!

The only required ingredients are the specification of the cross-section, and the standard formulae for the prolonged infinitesimal generators.

Theorem. If G acts transitively on M, or if the infinitesimal generator coefficients depend rationally in the coordinates, then all recurrence formulae are rational in the basic differential invariants and so $\mathcal{I}(G)$ is a rational, non-commutative differential algebra.

Generating Differential Invariants

As a consequence of the recurrence formulae:

Theorem. If the moving frame has order s, then the basic differential invariants $I^{(s+1)}$ of order $\leq\ s+1$ forms a generating set.

Theorem. If G acts transitively, then the Maurer–Cartan invariants form a generating set.

- Typically, these generating sets contain many redundancies, and are far from minimal.

The Extended Symbolic Invariant Calculus

However, in order to work in the polynomial category, we introduce symbolic variables to represent both the basic differential invariants and the Maurer–Cartan invariants.

The recurrence formulae then take the form

$$\widetilde{\mathcal{D}}_i \, v^\alpha_J = v^\alpha_{J,i} + \sum_{\kappa=1}^{r} w^\kappa_i \, \widetilde{\iota}\,(\varphi^\alpha_{J,\kappa})$$

$\widetilde{\mathcal{D}}_i$ — symbolic invariant differential operators, as defined by these formulas

$v^\alpha_J = \widetilde{\iota}\,(u^\alpha_J)$ — symbolic basic differential invariants

$\varphi^\alpha_{J,\kappa} = \mathbf{v}_\kappa(u^\alpha_J)$ — prolonged infinitesimal generator coefficients

w^κ_i — symbolic Maurer–Cartan invariants

$$\widetilde{\mathcal{D}}_j \, w^\kappa_{i;K} = w^\kappa_{i;j,K}$$

Functional Independence

$f_1(x^1, \ldots, x^m), \ldots, f_k(x^1, \ldots, x^m)$

— smooth scalar-valued functions.

Form their $k \times m$ *Jacobian matrix*:

$$\nabla f = \left(\frac{\partial f^i}{\partial x^j} \right).$$

Theorem. $f_1, \ldots, f_k$ are functionally independent if and only if their Jacobian matrix has maximal

$$\operatorname{rank} \nabla f = k$$

Restricted Functional Dependence

Suppose

$$M = \{\, x \in \mathbb{R}^n \mid\ c(x) = 0 \,\}$$

is a submanifold, where $c : \mathbb{R}^n \to \mathbb{R}^j$ and ∇c has constant rank in a neighborhood of M.

Given $f : \mathbb{R}^n \ \to\ \mathbb{R}^k$ and $g : \mathbb{R}^n \ \to\ \mathbb{R}^l$, then, locally, f is functionally dependent on g when restricted to M, so

$$f \mid M = h \circ g \mid M$$

for some $h : \mathbb{R}^l \to \mathbb{R}^k$, if and only if

$$\operatorname{rank}\begin{pmatrix} \nabla f \\ \nabla g \\ \nabla c \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \nabla g \\ \nabla c \end{pmatrix} \quad \text{on } M.$$

An Algorithm for
Finding (Minimal) Generating Sets

$I(v,w) = (\, I^1(v,w), \;\ldots\; , I^k(v,w)\,)$

— a known generating set of differential invariants

$J(v) = (\, J^1(v), \;\ldots\; , J^l(v)\,)$

— proposed generating set.

$J^{(n)}(v,w) = (\; \ldots \; J^\nu_K(v,w) = \widetilde{\mathcal{D}}_K J^\nu \; \ldots \;)$

— their symbolic derivatives up to level n

$0 = C^{(n)}(v,w) = (\; \ldots \; C^\sigma_{i;K}(v,w) \; \ldots \;)$ — differentiated cross-section equations $Z^\sigma = c^\sigma, \;\; \sigma = 1, \ldots, r$:

$$C^\sigma_{i;K} = \widetilde{\mathcal{D}}_K C^\sigma_i(v,w) = \widetilde{\mathcal{D}}_K \left(\widetilde{\iota}\,[\, D_i(Z^\sigma)\,] + \sum_{\kappa=1}^{r} \; w^\kappa_i \, \widetilde{\iota}\,[\, \mathbf{v}_\kappa(Z^\sigma)\,] \right)$$

Jacobian matrices:

$$\mathbb{J}^{(n)} = \begin{pmatrix} \nabla J^{(n)} \\ \nabla C^{(n)} \end{pmatrix}, \qquad \mathbb{I}^{(n)} = \begin{pmatrix} \nabla I \\ \nabla J^{(n)} \\ \nabla C^{(n)} \end{pmatrix}.$$

Theorem. The differential invariants $\{J^1, \ldots, J^l\}$ form a generating set if and only if

$$\operatorname{rank} \mathbb{I}^{(n)} = \operatorname{rank} \mathbb{J}^{(n)}$$

for some level $n \geq 0$.

The Algorithm

(1) Input the level n of the computation and the order k of the cross-section.

(2) Input the infinitesimal generators of the group action, and compute their prolongations up to order $n + k + 1$.

(3) Input the cross-section normalizations and check their validity.

(4) Compute the recurrence formulas up to order $n + k + 1$ in symbolic form.

(5) Compute the commutators in symbolic form.

(6) Compute the linear constraints induced by the cross-section up to level n.

(7) Choose a known generating set of differential invariants.

(8) Input the proposed generating differential invariants in symbolic form, and then compute their invariant derivatives up to level n.

(9) Compute the relevant Jacobian matrices. If the preceding rank condition is satisfied, then the chosen differential invariants form a generating set. If not, then either they are not generating, or one needs to go to a higher level n.

★★ To compute the ranks of the symbolic matrices in practice, substitute random integers for the variables they depend on, and compare the ranks of the corresponding integer matrices, repeating this computation several times to be sure. (Although repetition appears to be unnecessary.)

Thus, if successful, the algorithm will confirm that one has a generating set.

If unsuccessful, one can try a higher level.

Unfortunately, I do not know a bound on the level required to be sure whether or not the selected differential invariants are generating; this is a significant and apparently difficult open problem.

The Algebra of Euclidean Differential Invariants

Principal curvatures:

$$\kappa_1 = \iota(u_{xx}) \qquad \kappa_2 = \iota(u_{yy})$$

Mean curvature and Gauss curvature:

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2) \qquad K = \kappa_1 \kappa_2$$

Invariant differentiation operators:

$$\mathcal{D}_1 = \iota(D_x) \qquad \mathcal{D}_2 = \iota(D_y)$$

- Differentiation w.r.t. the diagonalizing Darboux frame.

The recurrence formulae enable one to express the higher order differential invariants in terms of the principal curvatures, or, equivalently, the mean and Gauss curvatures, and their invariant derivatives:

$$
\begin{aligned}
I_{jk} = \iota(u_{jk}) &= \widetilde{\Phi}_{jk}(\kappa_1, \kappa_2, \mathcal{D}_1\kappa_1, \mathcal{D}_2\kappa_1, \mathcal{D}_1\kappa_2, \mathcal{D}_2\kappa_2, \mathcal{D}_1^2\kappa_1, \ \ldots\ ) \\
&= \Phi_{jk}(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots\ )
\end{aligned}
$$

For H and for K:

level	size $\mathbb{J}^{(k)}$	rank $\mathbb{J}^{(k)}$	size $\mathbb{I}^{(k)}$	rank $\mathbb{I}^{(k)}$
0	13×18	13	14×18	14
1	39×47	39	40×47	40
2	91×101	91	92×101	92
3	195×204	195	196×204	195
4	403×404	394	404×404	394

The ranks are equal at level 3 so the level 4 computation is unnecessary, but was performed as a check on the algorithm.

Consequently, we can write K in terms of the third order invariant derivatives of H, which is thus generating, in accordance with a previously known result. Interestingly, the explicit formula that was found by direct manipulation of the recurrence formula involves the fourth order derivatives of H, and hence there is an as yet unknown formula for K involving at most third order derivatives of H.

$\star\star$ But this also says that K is generating, and so H can be expressed in terms of the third order invariant derivatives of K, which is a new and unexpected result.