Two New Developments for Noether's Two Theorems

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Part 1

Higher Order Symmetries of Underdetermined Systems of Partial Differential Equations

Symmetry Groups of Differential Equations

$\Longrightarrow$ Sophus Lie (1842–1899)

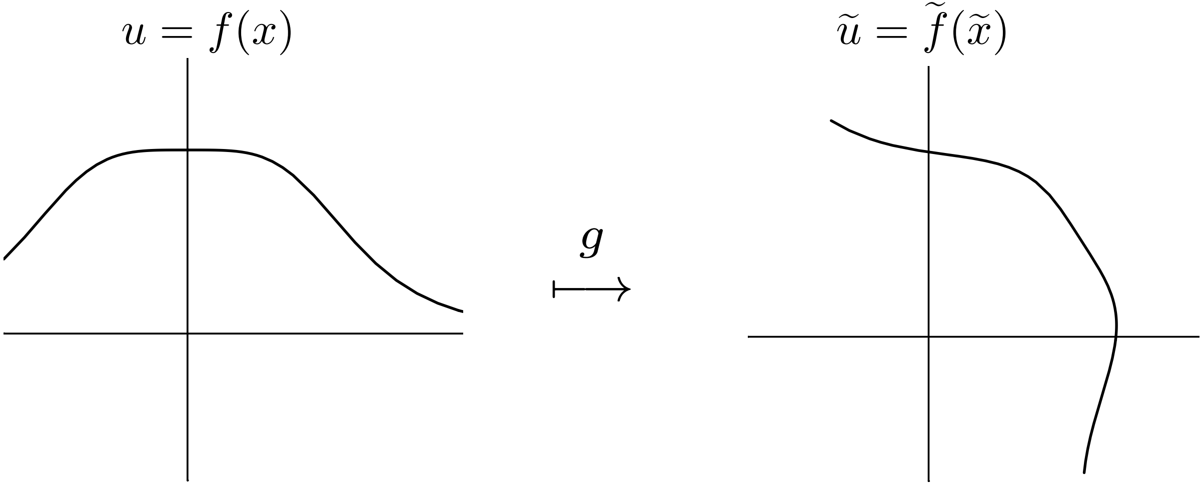
System of differential equations

$$\Delta(x, u^{(n)}) = 0$$

G — Lie group or Lie pseudo-group acting on the space of independent and dependent variables:

$$(\widetilde{x}, \widetilde{u}) = g \cdot (x, u)$$

G acts on functions by transforming their graphs:



Definition. G is a symmetry group of the system $\Delta = 0$ if $\widetilde{f} = g \cdot f$ is a solution whenever f is.

Infinitesimal Generators

Every one-parameter group can be viewed as the flow of a vector field $\mathbf{v}$, known as its infinitesimal generator.

In other words, the one-parameter group is realized as the solution to the system of ordinary differential equations governing the vector field's flow:

$$\frac{dz}{d\varepsilon} = \mathbf{v}(z)$$

Equivalently, if one expands the group transformations in powers of the group parameter ε, the infinitesimal generator comes from the linear terms:

$$z(\varepsilon) = z + \varepsilon\,\mathbf{v}(z) + \cdots$$

Infinitesimal Generators $=$ Vector Fields

In differential geometry, it has proven to be very useful to identify a vector field with a first order differential operator

In local coordinates $(\ldots x^i \ldots u^\alpha \ldots)$, the vector field

$$\mathbf{v} = (\ \ldots\ \xi^i(x,u)\ \ldots\ \varphi^\alpha(x,u)\ \ldots\ )$$

that generates the one-parameter group (flow)

$$\frac{dx^i}{d\varepsilon} = \xi^i(x,u) \qquad \frac{du^\alpha}{d\varepsilon} = \varphi^\alpha(x,u)$$

is identified with the differential operator

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

Prolongation

Since G acts on functions, it acts on their derivatives $u^{(n)}$, leading to the prolonged group action:

$$(\widetilde{x}, \widetilde{u}^{(n)}) = \mathrm{pr}^{(n)}\, g \cdot (x, u^{(n)})$$

$\Longrightarrow$ formulas provided by implicit differentiation

Prolonged infinitesimal generator:

$$\mathrm{pr}\, \mathbf{v} = \mathbf{v} + \sum_{\alpha, J} \varphi_J^\alpha(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha}$$

The Prolongation Formula

The coefficients of the prolonged vector field are given by the explicit prolongation formula (PJO, 1979):

$$\boxed{\varphi_J^\alpha = D_J\, Q^\alpha + \sum_{i=1}^{p} \xi^i\, u_{J,i}^\alpha}$$

where $D_i = \displaystyle\sum_{\alpha,J} u_{J,i}^\alpha \frac{\partial}{\partial u_J^\alpha}$ $\quad D_J = D_{j_1} \cdots D_{j_k}$ — total derivatives

$Q = (Q^1, \ldots, Q^q)$ — characteristic of $\mathbf{v}$

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i \frac{\partial u^\alpha}{\partial x^i}$$

★ Invariant functions $u = f(x)$ are solutions to $Q(x, u^{(1)}) = 0$

Lie's Infinitesimal Symmetry Criterion for Differential Equations

Theorem. A connected group of transformations G is a symmetry group of a nondegenerate system of differential equations $\Delta = 0$ if and only if

$$\boxed{\operatorname{pr}\mathbf{v}(\Delta) = 0 \qquad \text{whenever} \qquad \Delta = 0}$$

for every infinitesimal generator $\mathbf{v}$ of G.

Generalized (Higher Order) Symmetries

- ⋆ Due to Noether (1918)
- ⋆ *NOT* Lie or Bäcklund, who only got as far as contact transformations.

Key Idea: Allow the coefficients of the infinitesimal generator to depend on derivatives of u, but drop the requirement that the (prolonged) vector field define a geometrical transformation on any finite order jet space:

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u^{(k)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}$$

Characteristic :

$$Q_\alpha(x, u^{(k)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha$$

Evolutionary vector field:

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}$$

Prolongation formula:

$$\mathrm{pr}\, \mathbf{v} = \mathrm{pr}\, \mathbf{v}_Q + \sum_{i=1}^{p} \xi^i D_i$$

$$\mathrm{pr}\, \mathbf{v}_Q = \sum_{\alpha, J} D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha}$$

★ $\mathbf{v}$ is a generalized symmetry of a differential equation if and only if its evolutionary form $\mathbf{v}_Q$ is.

Example. Burgers' equation.

$$u_t = u_{xx} + u u_x$$

Characteristics of generalized symmetries:

$$u_x \qquad \text{space translations}$$

$$u_{xx} + u\,u_x \qquad \text{time translations}$$

$$u_{xxx} + \tfrac{3}{2}\,u\,u_{xx} + \tfrac{3}{2}\,u_x^2 + \tfrac{3}{4}\,u^2 u_x$$

$$u_{xxxx} + 2\,u\,u_{xxx} + 5\,u_x u_{xx} + \tfrac{3}{2}\,u^2 u_x + 3\,u\,u_x^2 + \tfrac{1}{2}\,u^3 u_x$$

$$\vdots$$

⟹ See Mikhailov–Shabat–Sokolov and J.P. Wang's thesis for long lists of equations with higher order symmetries.

Question: Which systems of PDE possess higher order generalized symmetries?

- Linear systems of partial differential equations that admit a nontrivial point symmetry group, as well as systems that can be linearized by a point or contact transformation or (in favorable circumstances) a differential substitution
  (C integrable systems)

- Integrable systems solvable by inverse scattering
  (S integrable systems)

★ Underdetermined systems that admit a symmetry generator depending on an arbitrary function of the independent variables

$\Longrightarrow$ Almost all equations with one higher order symmetry have infinitely many.

Bakirov's Counterexample:

The "triangular system" of evolution equations

$$u_t = u_{xxxx} + v^2 \qquad v_t = \tfrac{1}{5}\, v_{xxxx}$$

has one sixth order generalized symmetry, but no further higher order symmetries.

Bakirov (1991), Beukers–Sanders–Wang (1998),
van der Kamp–Sanders (2002)

★ ★ Non-triangular examples?

Recursion operators

$\Longrightarrow$ PJO (1977)

Definition. An operator $\mathcal{R}$ is called a recursion operator for the system $\Delta = 0$ if it maps symmetries to symmetries, i.e., if $\mathbf{v}_Q$ is a generalized symmetry (in evolutionary form), and $\widetilde{Q} = \mathcal{R}Q$, then $\mathbf{v}_{\widetilde{Q}}$ is also a generalized symmetry.

$\Longrightarrow$ A recursion operator generates infinitely many symmetries with characteristics

$$Q, \quad \mathcal{R}Q, \quad \mathcal{R}^2Q, \quad \mathcal{R}^3Q, \quad \ldots$$

Theorem. Given the system $\Delta = 0$ with Fréchet derivative (linearization) D_Δ, if

$$[D_\Delta, \mathcal{R}] = 0$$

on solutions, then $\mathcal{R}$ is a recursion operator.

Burgers' equation.

$$u_t = u_{xx} + uu_x \qquad \mathcal{R} = D_x + \tfrac{1}{2}u + \tfrac{1}{2}uD_x^{-1}$$

$$D_\Delta = D_t - D_x^2 - uD_x - u_x$$

$$D_\Delta \cdot \mathcal{R} = D_tD_x - D_x^3 - \tfrac{3}{2}uD_x^2 - \tfrac{1}{2}(5u_x + u^2)D_x + \tfrac{1}{2}u_t - \\ - \tfrac{3}{2}u_{xx} - \tfrac{3}{2}uu_x + \tfrac{1}{2}(u_{xt} - u_{xxx} - uu_{xx} - u_x^2)D_x^{-1},$$

$$\mathcal{R} \cdot D_\Delta = D_tD_x - D_x^3 - \tfrac{3}{2}uD_x^2 - \tfrac{1}{2}(5u_x + u^2)D_x - u_{xx} - uu_x$$

hence

$$[D_\Delta, \mathcal{R}] = \tfrac{1}{2}(u_t - u_{xx} - uu_x) + \tfrac{1}{2}(u_{xt} - u_{xxx} - uu_{xx} - u_x^2)D_x^{-1}$$

which vanishes on solutions.

Burgers' equation.

$$u_t = u_{xx} + u u_x$$

Recursion operator:

$$\mathcal{R} = D_x + \tfrac{1}{2}u + \tfrac{1}{2}u D_x^{-1}$$

Symmetries:

$$u_x$$

$$\mathcal{R}(u_x) = u_{xx} + u\,u_x$$

$$\mathcal{R}^2(u_x) = u_{xxx} + \tfrac{3}{2}\,u\,u_{xx} + \tfrac{3}{2}\,u_x^2 + \tfrac{3}{4}\,u^2 u_x$$

$$\mathcal{R}^3(u_x) = u_{xxxx} + 2\,u\,u_{xxx} + 5\,u_x u_{xx} + \tfrac{3}{2}\,u^2 u_x + 3\,u\,u_x^2 + \tfrac{1}{2}\,u^3 u_x$$

$$\vdots$$

Linear Equations

Theorem. Let

$$\Delta[u] = 0$$

be a linear system of partial differential equations. Then any symmetry $\mathbf{v}_Q$ with linear characteristic $Q = \mathcal{D}[u]$ determines a recursion operator $\mathcal{D}$, since

$$[\mathcal{D}, \Delta] = \widetilde{\mathcal{D}} \cdot \Delta$$

If $\mathcal{D}_1, \ldots, \mathcal{D}_m$ determine linear symmetries $\mathbf{v}_{Q_1}, \ldots, \mathbf{v}_{Q_m}$, then any polynomial in the $\mathcal{D}_j$'s also gives a linear symmetry.

Question 1: Given a linear system, when are all symmetries
a) linear? b) generated by first order symmetries?

Question 2: What is the structure of the non-commutative symmetry algebra?

Bi–Hamiltonian systems

$\Longrightarrow$ Magri (1978)

Theorem. Suppose

$$\frac{\partial u}{\partial t} = F_1 = J_1 \frac{\delta \mathcal{H}_1}{\delta u} = J_2 \frac{\delta \mathcal{H}_0}{\delta u}$$

is a biHamiltonian system, where J_1, J_2 form a compatible pair of Hamiltonian operators. Assume that J_1 is nondegenerate. Then

$$\mathcal{R} = J_2 J_1^{-1}$$

is a recursion operator that generates an infinite hierarchy of biHamiltonian symmetries

$$\frac{\partial u}{\partial t} = F_n = \mathcal{R}\, F_{n-1} = J_1 \frac{\delta \mathcal{H}_n}{\delta u} = J_2 \frac{\delta \mathcal{H}_{n-1}}{\delta u}$$

The Korteweg–deVries Equation

$$\frac{\partial u}{\partial t} = u_{xxx} + u\,u_x = J_1\,\frac{\delta \mathcal{H}_1}{\delta u} = J_2\,\frac{\delta \mathcal{H}_0}{\delta u}$$

$$J_1 = D_x \qquad \mathcal{H}_1[u] = \int \left(\tfrac{1}{6}\,u^3 - \tfrac{1}{2}\,u_x^2 \right) dx$$

$$J_2 = D_x^3 + \tfrac{2}{3} u\,D_x + \tfrac{1}{3}\,u_x \qquad \mathcal{H}_0[u] = \int \tfrac{1}{2}\,u^2\,dx$$

★★ Bi–Hamiltonian system with recursion operator (Lenard)

$$\mathcal{R} = J_2 \cdot J_1^{-1} = D_x^2 + \tfrac{2}{3}\,u + \tfrac{1}{3}\,u_x\,D_x^{-1}$$

Hierarchy of generalized symmetries and higher order conservation laws:

$$\frac{\partial u}{\partial t} = u_{xxxxx} + \tfrac{5}{3}\,u\,u_{xxx} + \tfrac{10}{3}\,u_x u_{xx} + \tfrac{5}{6}\,u^2 u_x = J_1\,\frac{\delta \mathcal{H}_2}{\delta u} = J_2\,\frac{\delta \mathcal{H}_1}{\delta u}$$

$$\mathcal{H}_2[u] = \int \left(\tfrac{1}{2}\,u_{xx}^2 - \tfrac{5}{6}\,u_x^2 + \tfrac{5}{72}\,u^4 \right) dx$$

and so on ... (Gardner, Green, Kruskal, Miura, Lax)

Underdetermined Systems

We assume that the system of differential equations

$$\Delta_\kappa[u] = 0, \quad \kappa = 1, \ldots, q, \qquad (*)$$

has the same number of equations as unknowns $u = (u^1, \ldots, u^q)$.

Definition. The system of differential equations $(*)$ is *underdetermined* if there exist differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_q$ that do not simultaneously vanish on solutions, such that

$$\mathcal{D}_1 \Delta_1 + \cdots + \mathcal{D}_q \Delta_q \equiv 0.$$

For the general case (which is quite subtle) see

W.M. Seiler, *Involution*, Springer, 2010.

Examples of underdetermined systems arising in basic physics include Maxwell's equations for electromagnetism and Einstein's equations for general relativity.

The Main Theorem

Theorem. Suppose that a system of differential equations admits an infinitesimal symmetry $\mathbf{v}_Q$ whose characteristic

$$Q[u,h] = Q(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ h_K(x)\ \ldots\ )$$

depends on finitely many derivatives $h_K = \partial_K h$ of an arbitrary function $h(x)$ of the independent variables. Let $F[u]$ be an arbitrary differential function. Then the characteristic

$$\widehat{Q}[u] = Q(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ D_K F\ \ldots\ )$$

obtained by replacing the derivatives of h by the corresponding total derivatives of F is also the characteristic of an infinitesimal symmetry $\mathbf{v}_{\widehat{Q}}$ of the system.

Thus, any such underdetermined system of differential equations automatically admits an infinite family of higher order symmetries depending upon an arbitrary function F of the independent variables, the dependent variables, and their derivatives of arbitrarily high order.

⋆⋆ Systems that satisfy the hypothesis of the Theorem are necessarily underdetermined, although, as we will see, not every underdetermined system will admit such a symmetry generator.

Proof:

First: note that the partial derivatives of h coincide with its total derivatives: $\partial_K h(x) = D_K h$.

Second: Suppose

$$R(\ \dots\ x^i\ \dots\ u^\alpha_J\ \dots\ \partial_K h(x)\ \dots\ ) = 0$$

where $h(x)$ is an arbitrary function of all the independent variables. Then, since its partial derivatives $\partial_K h(x)$ can assume any values, we can replace them by any quantities and still have equality. In particular,

$$R(\ \dots\ x^i\ \dots\ u^\alpha_J\ \dots\ D_K F\ \dots\ ) = 0$$

where $F[u]$ be an arbitrary differential function.

$\Longrightarrow$ Kiselev's Substitution Principle:

`https://preprints.ihes.fr/2012/M/M-12-13.pdf`

Third: according to the prolongation formula for evolutionary vector fields, the coefficients of $\mathrm{pr}\,\mathbf{v}_Q$ are obtained by total differentiation, so

$$D_I Q_\alpha = R_{\alpha,I}(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ \partial_K h(x)\ \ldots\ ),$$

where $R_{\alpha,I}$ are certain functions of the jet coordinates and the partial (total) derivatives of h, then, replacing h by F in Q leads, via the substitution principle, to the same algebraic expressions for its total derivatives

$$D_I \widehat{Q}_\alpha = R_{\alpha,I}(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ D_K F\ \ldots\ ),$$

in terms of the jet coordinates and the total derivatives of F. By the preceding remarks, we can thus replace each partial derivative $h_K(x)$ appearing in the determining equations by the corresponding total derivative $D_K F$ without affecting their validity. We conclude that the evolutionary vector field $\mathbf{v}_{\widehat{Q}}$ with characteristic $\widehat{Q}$ also satisfies the symmetry determining equations for the system of differential equations. *Q.E.D.*

Variational Symmetries

Definition. A (strict) variational symmetry is a transformation $(\widetilde{x}, \widetilde{u}) = g \cdot (x, u)$ which leaves the variational problem invariant:

$$\int_{\widetilde{\Omega}} L(\widetilde{x}, \widetilde{u}^{(n)})\, d\widetilde{x} = \int_{\Omega} L(x, u^{(n)})\, dx$$

Infinitesimal invariance criterion:

$$\operatorname{pr} \mathbf{v}(L) + L \operatorname{Div} \xi = 0$$

Divergence symmetry (Bessel–Hagen):

$$\operatorname{pr} \mathbf{v}(L) + L \operatorname{Div} \xi = \operatorname{Div} B$$

$\Longrightarrow$ Every divergence symmetry has an equivalent strict variational symmetry

Theorem. Every symmetry of a variational problem is a symmetry of the Euler–Lagrange equations.

★ ★ But not conversely!

★ Almost all examples of non-variational symmetries are scaling symmetries. One known exception is the equations of 3D linear isotropic elasticity which admits the non-variational generalized symmetry whose flow is equivalent to Maxwell's equations! (PJO, 1984)

Noether's Second Theorem

Theorem. A system of Euler-Lagrange equations is underdetermined if and only if it admits an infinite dimensional variational symmetry group depending on an arbitrary function of the independent variables. The associated conservation laws are trivial.

★ Noether's First Theorem gives a one-to-one correspondence between non-trivial symmetries and non-trivial conservation laws. (Martinez Alonso, 1979; PJO, 1986)

Open Question: Are there over-determined systems of Euler–Lagrange equations for which trivial symmetries give non-trivial conservation laws?

Generalized Noether's Second Theorem

Theorem. If $E(L) = 0$ is any underdetermined system of Euler–Lagrange equations, then it admits generalized symmetries of arbitrarily high order depending upon one or more arbitrary differential functions.

This result resolves a mystery concerning Noether's Second Theorem, which relies on infinitesimal symmetries that involve one or more arbitrary functions of the p independent variables. But what about the functions of the dependent variables, and hodograph and reciprocal transformations that interchange independent and dependent variables, etc.? According to the Theorem, once the system admits variational symmetries depending on an arbitrary function of *any* p jet variables, then it automatically admits variational symmetries depending on an arbitrary differential function!

Relativity

Noether's Second Theorem effectively resolved Hilbert's dilemma regarding the law of conservation of energy in Einstein's field equations for general relativity.

Namely, the time translational symmetry that ordinarily leads to conservation of energy in fact belongs to an infinite-dimensional symmetry group, and thus, by Noether's Second Theorem, the corresponding conservation law is trivial, meaning that it vanishes on all solutions.

$\Longrightarrow$ Higher order symmetries of Einstein's equations:

Anderson and Torre, 1993

A Simple Example:

Variational problem:

$$I[u,v] = \iint (u_x + v_y)^2 \, dx \, dy$$

Variational symmetry group:

$$(u,v) \longmapsto (u + \varphi_y, v - \varphi_x)$$

Evolutionary generator:

$$\mathbf{v}_Q = -\frac{\partial h}{\partial y}\frac{\partial}{\partial u} + \frac{\partial h}{\partial x}\frac{\partial}{\partial v}$$

Euler-Lagrange equations:

$$\Delta_1 = E_u(L) = u_{xx} + v_{xy} = 0$$

$$\Delta_2 = E_v(L) = u_{xy} + v_{yy} = 0$$

Differential relation:

$$D_y \Delta_1 - D_x \Delta_2 \equiv 0$$

The Main Theorem implies that, for any differential function $F[u, v]$ depending on x, y and u, v and their derivatives, the evolutionary vector field

$$\widehat{\mathbf{v}} = -D_y F \frac{\partial}{\partial u} + D_x F \frac{\partial}{\partial v}$$

is also a variational symmetry, and thus a higher order symmetry of the underdetermined Euler–Lagrange equations.

For example, the second order variational problem

$$\widetilde{I}[u,v] = \iint \left[\tfrac{1}{2}(u_{xx} + v_{xy})(u_{xy} + v_{yy}) + \tfrac{1}{6}(u_x + v_y)^3 \right] dx\, dy,$$

with underdetermined nonlinear fourth order Euler–Lagrange equations

$$u_{xxxy} + v_{xxyy} = (u_x + v_y)(u_{xx} + v_{xy}),$$

$$u_{xxyy} + v_{xyyy} = (u_x + v_y)(u_{xy} + v_{yy}),$$

possesses the aforementioned properties.

Systems of differential equations or variational problems for curves, surfaces, etc., that do not depend on any underlying parametrization thereof are called *parameter-independent*. The symmetry pseudo-group consisting of all local diffeomorphisms of the base space X has infinitesimal generators

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x) \frac{\partial}{\partial x^i},$$

where $\xi^i(x)$ are arbitrary functions.

Theorem. A system of differential equations $\Delta[u] = 0$ is parameter-independent if and only if it admits all generalized infinitesimal symmetry generators of the form

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} \left(\sum_{i=1}^{p} u_i^\alpha F_i[u] \frac{\partial}{\partial u^\alpha} \right),$$

where $F_1[u], \ldots, F_p[u]$ are arbitrary differential functions.

Part 2

Divergence Invariant Variational Problems

One of the most remarkable methodological trends in modern physics has been the a priori *use of symmetry principles to constrain the action principles of the non-gravitational interactions in quantum electrodynamics (QED) and particle physics.*

— Harvey Brown

The Modern Manual for Physics

as envisioned by E. Noether (1918)

As Hilbert expresses his assertion, the lack of a proper law of energy constitutes a characteristic of the "general theory of relativity." For that assertion to be literally valid, it is necessary to understand the term "general relativity" in a wider sense than is usual, and to extend it to the aforementioned groups that depend on n arbitrary functions.[27]

[27] *This confirms once more the accuracy of Klein's remark that the term "relativity" as it is used in physics should be replaced by "invariance with respect to a group."*

The Modern Manual for Physics

♠ To construct a physical theory:

Step 1: Determine the allowed group of symmetries:

- translations
- rotations
- conformal (angle-preserving) transformations
- Galilean boosts
- Poincaré transformations: $\mathrm{SO}(4,2)$ (special relativity)
- gauge transformations
- CPT (charge, parity, time reversal) symmetry
- supersymmetry
- $\mathrm{SU}(3)$, G_2, $\mathrm{E}_8 \times \mathrm{E}_8$, $\mathrm{SO}(32)$, $\ldots$
- etc., etc.

Step 2: Construct a variational principle ("energy") that admits the given symmetry group.

Step 3: Invoke Nature's obsession with minimization to construct the corresponding field equations (Euler–Lagrange equations) associated with the variational principle.

Step 4: Use Noether's First and Second Theorems to write down (*a*) conservation laws, and (*b*) differential identities satisfied by the field equations.

Step 5: Try to solve the field equations.

Even special solutions are of immense interest

$\Longrightarrow$ black holes.

All Known Physics

$$\Psi = \int e^{\frac{i}{\hbar}\int\left(\frac{R}{16\pi G} - \frac{1}{4}F^2 + \overline{\psi} i \not{D} \psi - \lambda\varphi\overline{\psi}\psi + |D\varphi|^2 - V(\varphi)\right)}$$

Schrödinger
Feynman
Einstein
Maxwell
Kobayashi-
Maskawa
Lagrange
Euler
Planck
Newton
Dirac
Yukawa
Higgs

$\Longrightarrow$ Neil Turok (Perimeter Institute)

Characterization of Invariant Variational Problems

According to Lie, any strictly invariant variational problem can be written in terms of the differential invariants:

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\,\ldots\ \mathcal{D}_K I^\alpha\ \ldots\,)\, \boldsymbol{\omega}}$$

$I^1, \ldots, I^\ell$ — fundamental differential invariants

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ — invariant differential operators

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — invariant volume form

If the variational problem is G-invariant, so

$$\mathcal{I}[u] = \int P(\,\ldots\, \mathcal{D}_K I^\alpha \,\ldots\,)\, \boldsymbol{\omega}$$

then its Euler–Lagrange equations admit G as a symmetry group, and hence can also be expressed in terms of the differential invariants:

$$E(L) \simeq F(\,\ldots\, \mathcal{D}_K I^\alpha \,\ldots\,) = 0$$

Problem: Construct F directly from P.

$\Longrightarrow$ Solved in general by Irina Kogan & PJO (2001) using moving frames

A Physical Conundrum

Since all Lie groups and most Lie pseudo-groups admit infinitely many differential invariants, there are an infinite number of distinct invariant variational principles

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\ \boldsymbol{\omega}}$$

★ ★ Physicists are extraordinarily talented at finding the "simplest" such invariant variational principle, even for very complicated physical symmetry groups. Such a principle then forms the basis of fundamental physics.

A Physical Conundrum

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\ \boldsymbol{\omega}}$$

On the other hand, physicists seem to be mostly unaware of the theory of differential invariants and the consequent existence of infinitely many alternative invariant variational principles, hence:

Does the underlying physics depend upon which of these invariant variational principles is used and, if so, how does one select the "correct" physical variational principle?

Variational Symmetries

Definition. A strict variational symmetry is a transformation $(\widetilde{x}, \widetilde{u}) = g \cdot (x, u)$ which leaves the variational problem invariant:

$$\int_{\widetilde{\Omega}} L(\widetilde{x}, \widetilde{u}^{(n)})\, d\widetilde{x} = \int_{\Omega} L(x, u^{(n)})\, dx$$

Infinitesimal invariance criterion:

$$\operatorname{pr} \mathbf{v}(L) + L \operatorname{Div} \xi = 0$$

Divergence symmetry (Bessel–Hagen):

$$\operatorname{pr} \mathbf{v}(L) + L \operatorname{Div} \xi = \operatorname{Div} B$$

$\Longrightarrow$ Every divergence symmetry has an equivalent strict variational symmetry

Noether's First Theorem

Theorem. If $\mathbf{v}$ generates a one-parameter group of variational symmetries of a variational problem, then the characteristic Q of $\mathbf{v}$ is the characteristic of a conservation law of the Euler-Lagrange equations:

$$\operatorname{Div} P = Q\, E(L)$$

Noether's Example

★ ★ See Noether, p. 245.

The one-dimensional free particle (unit mass)

$$I[u] = \int \tfrac{1}{2} u_t^2 \, dt,$$

Euler–Lagrange equation:

$$u_{tt} = 0.$$

One-parameter group of Galilean symmetries

$$(t, u) \longmapsto (t, u + \varepsilon\, t), \qquad \varepsilon \in \mathbb{R}.$$

Prolonged action:

$$u_t \longmapsto u_t + \varepsilon, \qquad u_{tt} \longmapsto u_{tt}, \qquad \ldots\ .$$

$$I[u] = \int \tfrac{1}{2} u_t^2 \, dt$$

$$(t, u, u_t, u_{tt}, \ \ldots\ ) \ \longmapsto \ (t, u + \varepsilon\, t, u_t + \varepsilon, u_{tt}, \ \ldots\ )$$

The Lagrangian is divergence invariant since

$$\begin{aligned} \tfrac{1}{2} u_t^2 \ \longmapsto \ \tfrac{1}{2}(u_t + \varepsilon)^2 &= \tfrac{1}{2} u_t^2 + \varepsilon\, u_t + \tfrac{1}{2}\varepsilon^2 \\ &= \tfrac{1}{2} u_t^2 + D_t(\, \varepsilon\, u + \tfrac{1}{2}\varepsilon^2 t\,). \end{aligned}$$

Noether conservation law:

$$D_t(t\, u_t - u) = t\, u_{tt} = 0.$$

$$(t, u, u_t, u_{tt}, \ \ldots\ ) \longmapsto (t, u + \varepsilon\, t, u_t + \varepsilon, u_{tt}, \ \ldots\ )$$

Differential invariants:

$$t, \quad v = u_t - \frac{u}{t}, \quad u_{tt}, \quad u_{ttt}, \quad \ldots\ .$$

Invariantized variational principle:

$$J[\,u\,] = \int \tfrac{1}{2} v^2 \, dt = \int \tfrac{1}{2} (u_t - u/t)^2 \, dt,$$

★ ★ The invariantized Lagrangian is strictly Galilean invariant.

Since

$$\frac{1}{2}\left(u_t - \frac{u}{t}\right)^2 = \tfrac{1}{2} u_t^2 - \frac{u\, u_t}{t} + \frac{u^2}{2\,t^2} = \tfrac{1}{2} u_t^2 - D_t \left(\frac{u^2}{2\,t}\right),$$

the variational problems are equivalent and have the same free particle Euler–Lagrange equation.

According to Lie's Theorem, the most general strictly Galilean invariant variational problem has the form

$$K[u] = \int F(t, v, u_{tt}, u_{ttt}, \ \ldots \ , u_{n\,t})\, dt$$

$$= \int F(t, u_t - u/t, u_{tt}, u_{ttt}, \ \ldots \ , u_{n\,t})\, dt$$

Suppose $F = F(v)$. The Euler–Lagrange equation is

$$-F''(u_t - u/t)\, u_{tt} + \frac{B(u_t - u/t)}{t} = 0,$$

where

$$B(v) = v\, F''(v) - F'(v).$$

We can integrate once:

$$\int \frac{F''(v)}{F'(v)}\, dv = -\int \frac{dt}{t}, \quad \text{and hence} \quad F'(v) = \frac{1}{c\, t}, \quad c \in \mathbb{R}.$$

$$F'(v) = F'\left(u_t - \frac{u}{t}\right) = \frac{1}{c\,t}, \quad c \in \mathbb{R}.$$

Solving for

$$v = u_t - \frac{u}{t} = G(c\,t), \quad \text{where} \quad G(x) = F'^{-1}(1/x).$$

the resulting first order ordinary differential equation is linear, and its general solution is nonlinear in time:

$$u(t) = t\,(a + H(c\,t)), \qquad \text{where} \qquad H'(x) = \frac{G(x)}{x}, \quad a \in \mathbb{R}.$$

In particular, for Noether's variational principle, $F(v) = \frac{1}{2}v^2$, so that $G(x) = 1/x,\ \ H(x) = -1/x$, and so

$$u(t) = a\,t - 1/c = a\,t + b,$$

recovering the standard linear motion of a free particle.

★★ Note that the nonlinear motion induced by the general Galilean-invariant Lagrangian is mathematically quite different from the linear free particle motion, and there is no obvious relationship between the two,
neither mathematical nor physical.

Full Galilean group

$$(t, u) \longmapsto (t + a, u + \varepsilon\, t + b), \qquad a, b, \varepsilon \in \mathbb{R}.$$

⋆⋆ The free particle Lagrangian is strictly invariant under the translations, but divergence invariant under the Galilean boost. On the other hand, Noether's Lagrangian is strictly invariant under the Galilean boost, but only divergence invariant under the translations.

The conservation laws corresponding to the translation symmetries are momentum and energy:

$$D_t(u_t) = u_{tt} = 0, \qquad D_t\left(\tfrac{1}{2}\, u_t^2\right) = u_t u_{tt} = 0.$$

Differential invariants: $u_{tt}, u_{ttt}, \ \ldots$

Strictly Galilean-invariant variational problem:

$$\widetilde{K}[u] = \int F(u_{tt}, u_{ttt}, \ \ldots \ , u_{n\,t})\, dt.$$

There are no non-constant strictly Galilean-invariant first order Lagrangians!

Divergence-invariant Lagrangians:

$$\widehat{K}[u] = \int \left[\tfrac{1}{2} m\, u_t^2 + f\, u + F(u_{tt}, u_{ttt}, \ \ldots \ , u_{n\,t}) \right] dt,$$

where $m \in \mathbb{R}$ is mass and $f \in \mathbb{R}$ a uniform external force.

If F depends nonlinearly on the n^{th} order derivative $u_{n\,t}$, the corresponding Euler–Lagrange equation has order $2n$, and its solutions appear to have very little to do with physical free particle motion.

Divergence-invariant first order Lagrangians:

$$\widehat{K}_1[u] = \int \left[\, \tfrac{1}{2} m\, u_t^2 + f\, u \,\right] dt.$$

Galilean-invariant Euler–Lagrange equations:

$$m\, u_{tt} = f, \qquad \text{with solution} \qquad u(t) = \frac{f\, t^2}{2\, m} + a\, t + b.$$

Noether conservation laws:

$$D_t(m\, u_t - f\, t) = m\, u_{tt} - f = 0,$$

$$D_t \left(\tfrac{1}{2} m\, u_t^2 - f\, u \right) = u_t (m\, u_{tt} - f) = 0,$$

$$D_t \left(m\, (t\, u_t - u) - \tfrac{1}{2} f\, t^2 \right) = t\, (m\, u_{tt} - f) = 0.$$

★★ momentum, energy, and quadratic time dependence of the particle's motion.

General Considerations

Let $\mathbf{v}$ be a generalized vector field and $\mathbf{v}_Q$ its evolutionary representative.

Divergence invariance under $\mathbf{v}$ if and only if $\mathbf{v}_Q$ is also a variational symmetry and hence satisfies

$$E[\,\mathrm{pr}\,\mathbf{v}_Q(L)\,] = \mathrm{pr}\,\mathbf{v}_Q[\,E(L)\,] + D_Q^* E(L) = 0,$$

where D_Q^* denotes the adjoint of the Fréchet derivative of Q.

$\mathbf{v}$ — distinguished symmetry of the Euler–Lagrange equations.

Lemma. A Lagrangian L is divergence invariant under $\mathbf{v}$ if and only if $\mathbf{v}$ is a distinguished symmetry of the Euler–Lagrange equations $E(L) = 0$.

The Euler–Lagrange Complex

$p = \#$ independent variables

$$\mathbb{R} \longrightarrow \Omega^0 \xrightarrow{\text{Grad}} \Omega^1 \longrightarrow \cdots$$

$$\cdots \longrightarrow \Omega^{p-1} \xrightarrow{\text{Div}} \Omega^p \xrightarrow{E} \mathcal{F}^1 \xrightarrow{\delta} \mathcal{F}^2 \longrightarrow \cdots$$

$\Longrightarrow$ Total deRham complex

Ω^{p-1} — Conservation laws

Ω^p — Lagrangians

$\mathcal{F}^1$ — Source forms (PDEs)

$\mathcal{F}^2$ — Helmholtz conditions

$\star\star$ The Euler–Lagrange complex is exact.

The Invariant Euler–Lagrange Complex

A differential form ω is $\mathfrak{g}$-invariant if and only if

$$\mathbf{v}(\omega) = 0 \qquad \text{for all} \qquad \mathbf{v} \in \mathfrak{g}.$$

$\star\star$ The $\mathfrak{g}$-invariant Euler–Lagrange complex is not exact — there can be nontrivial cohomology.

Theorem. (Anderson-Pohjanpelto, Thompson-Valiquette) If $\dim \mathfrak{g} < \infty$, then the cohomology of the $\mathfrak{g}$-invariant Euler–Lagrange complex is isomorphic to the Lie algebra cohomology $H^*(\mathfrak{g})$.

Divergence Invariance and Cohomology

- ★ A Lagrangian form is $\mathfrak{g}$-invariant if and only if the variational problem is strictly invariant
- ★ The Euler–Lagrange source form is $\mathfrak{g}$-invariant if and only if $\mathfrak{g}$ is a distinguished symmetry algebra, and hence the Lagrangian form is divergence invariant.

Theorem. A divergence $\mathfrak{g}$-invariant Lagrangian form λ is equivalent to a strictly $\mathfrak{g}$-invariant Lagrangian form if and only if the cohomology class of its Euler–Lagrange source form vanishes:

$$0 = [\,E(\lambda)\,] \in H^{p+1}(\mathfrak{g}).$$

Thus, $H^{p+1}(\mathfrak{g}) = \{0\}$ is necessary and sufficient for every divergence-invariant Lagrangian to be strictly invariant.

If $\dim \mathfrak{g} \leq p$, then $H^{p+1}(\mathfrak{g}) = \{0\}$ automatically.

In Noether's example, there is one independent variable t, and hence $p = 1$. Thus, in this situation, Noether's "trick" of replacing the free particle Lagrangian with an equivalent strictly invariant Lagrangian can be applied to *any* one-parameter symmetry group, and hence divergence inequivalence is a multi-parameter phenomenon.

Example. Translations

$$\mathfrak{g}_2 : \qquad (x,u) \longmapsto (x+a, u+b)$$

Differential invariants:

$$u_t, u_{tt}, u_{ttt}, \ldots$$

Lie algebra cohomology:

$$H^2(\mathfrak{g}_2) = \mathbb{R}$$

Divergence invariant variational problem

$$I[u] = \int \left[\, f\,u + F(u_t, u_{tt}, \ \ldots \ , u_{n\,t}) \,\right] dt,$$

where $f \in \mathbb{R}$ and the second term is strictly invariant.

Example. Galilean group plus time and space scalings:

$$\mathfrak{g}_5 : \qquad (t, u) \longmapsto (\lambda t + a, \mu u + \varepsilon t + b)$$

Differential invariants:

$$I_4 = \frac{u_{tt} u_{tttt}}{u_{ttt}^2}, \qquad I_5 = \frac{u_{tt}^2 u_{ttttt}}{u_{ttt}^3}, \qquad \ldots \qquad I_n = \frac{u_{tt}^{n-3} u_{n\,t}}{u_{ttt}^{n-2}},$$

Strictly invariant variational problem:

$$I[u] = \int F(I_4, I_5, \ \ldots \ , I_n) \, \frac{u_{ttt}}{u_{tt}} \, dt$$

$$\mathfrak{g}_5 : \quad (t, u) \longmapsto (\lambda t + a, \mu u + \varepsilon t + b)$$

Lie algebra cohomology: $H^2(\mathfrak{g}_5) = \mathbb{R}$

Divergence (but not strictly) invariant variational problem:

$$J[u] = \int \frac{u_{ttt} \log u_{ttt} - u_{ttt}}{u_{tt}} \, dt,$$

Euler–Lagrange equation:

$$D_t^2 \left(\frac{u_{tt} u_{tttt} - u_{ttt}^2}{u_{tt}^2 u_{ttt}} \right) = 0.$$

Final Remarks on the Foundations of Physics

Let us close with a wild speculation that our Theorem may provide an answer to the original question. Namely, we propose that variational principles of physical relevance are distinguished by arising from nonzero cohomology classes of the underlying physical symmetry group, or, equivalently, are based on divergence invariant Lagrangians that are not equivalent to any strictly invariant Lagrangian.

In particular, if the relevant cohomology space is one-dimensional, such a variational principle is unique up to constant multiple.

Thus, divergence invariance or, equivalently, cohomological considerations may be fundamental to the symmetry-driven formulation of physical theories.

It would be of great interest to determine the cohomology for the infinite-dimensional group underlying the standard model, although this will be a very challenging computation.

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