

# Affine Invariant Gradient Flows\*

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**Summary.** An affine invariant metric allowing one to compute affine invariant gradient descent flows is first presented in this work. This means that given an affine invariant energy, we compute based on this metric the flow that minimizes this energy as fast as possible and in an affine invariant way. Two examples are then presented. The first one shows that the affine flow minimizing the area enclosed by a planar curve is given by the affine geometric heat flow. We then extend this energy to derive affine invariant active contours for invariant image segmentation.

## 1. Introduction

A number of problems in image processing and computer vision are approached via energy minimization techniques. The minimizer can be obtained for example via gradient descent flows, which are flows minimizing this energy as fast as possible according to certain metric. In a number of cases, as for example in object recognition, the energy being minimized is invariant to certain transformation group, and the solution is expected to be invariant as well (see for example [2]). In order to obtain a gradient flow which is also invariant, an invariant metric needs to be defined.

In this work, we present an affine invariant metric which will help us to define affine invariant gradient descent flows for affine invariant energies. Two applications of this gradient flow are then provided. The first one shows that the affine invariant gradient flow corresponding to the area enclosed by a planar curve is given by the affine geometric heat flow introduced in [19, 20] (see also [1]). We then extend this area and present affine invariant active contours, extending the results in [5, 11]. In this case, active contours (introduced by Terzopoulos *et al.* [10, 22]) are given by an affine invariant weighted distance [16]. The affine gradient flow minimizing this distance converges to the objects boundaries, obtaining affine invariant detection.

## 2. Planar curve evolution

The theory of planar curve evolution has been considered in a large variety of fields. See [18] and references in there for pointers to some of the relevant

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literature. Formally, let  $\mathcal{C}(p, t) : S^1 \times [0, \tau) \rightarrow \mathbb{R}^2$  be a family of smooth embedded closed curves in the plane (boundaries of planar shapes). Assume that this family of curves evolves according to  $(\mathcal{C}(p, 0) = \mathcal{C}_0(p))$

$$\frac{\partial \mathcal{C}(p, t)}{\partial t} = \frac{\partial^2 \mathcal{C}(p, t)}{\partial v^2} = \kappa(p, t) \mathcal{N}(p, t). \quad (2.1)$$

Here  $v(p) = \int_0^p \|\mathcal{C}_p\| dp$  is the *Euclidean arc-length* ( $\|\mathcal{C}_v\| \equiv 1$ ),  $\kappa = [\mathcal{C}_v \times \mathcal{C}_{vv}]$  the *Euclidean curvature*, and  $\mathcal{N}$  the *inward unit normal*. The flow given by (2.1) is called the *Euclidean shortening flow*, since the curve perimeter shrinks as fast as possible when the curve evolves according to it [9]. Gage and Hamilton [8] and Grayson [9] proved that any embedded curve in the plane converges to a round point via the flow given in (2.1). The non-linear flow (2.1) is also called the *Euclidean geometric heat flow*. It has been utilized for the definition of a geometric, Euclidean invariant, multiscale representation of planar shapes [1, 12].

Recently, we introduced a new curve evolution equation, the *affine geometric heat flow* [19, 20]:

$$\frac{\partial \mathcal{C}(p, t)}{\partial t} = \frac{\partial^2 \mathcal{C}(p, t)}{\partial s^2}, \quad (2.2)$$

where  $s(p) = \int_0^p [\mathcal{C}_p \times \mathcal{C}_{pp}]^{1/3} dp$ , is the *affine arc-length* ( $[\mathcal{C}_s \times \mathcal{C}_{ss}] \equiv 1$ ) [3], and  $\mathcal{C}_{ss}$  is the *affine normal*. This evolution is the affine analogue of equation (2.1), and its solution space is affine invariant. Since the affine normal  $\mathcal{C}_{ss}$  exists just for non-inflection points, we formulated the natural extension of the flow (2.3) for non-convex initial curves in [20, 21]:

$$\frac{\partial \mathcal{C}(p, t)}{\partial t} = \begin{cases} 0, & p \text{ an inflection point,} \\ \mathcal{C}_{ss}(p, t), & \text{otherwise.} \end{cases} \quad (2.3)$$

The flow (2.3) defines a geometric, affine invariant, multiscale representation of planar shapes [20]. The curve first becomes convex, as in the Euclidean case, and after that it converges into an ellipse [19].

We should also add that in [21], we give a general method for writing down invariant flows with respect to any Lie group action on  $\mathbb{R}^2$ . This was formalized and extended to  $\mathbb{R}^n$ , together with uniqueness results, in [14, 15]. Results for the projective group were recently also reported in [7].

### 3. Affine invariant curve metric

Let  $\mathcal{C} = \mathcal{C}(p, t)$  be a smooth family of closed curves where  $t$  parametrizes the family and  $p$  the given curve, say  $0 \leq p \leq 1$ . Consider the length functional  $L(t) := \int_0^1 \|\mathcal{C}_p\| dp$ , and its first variation  $L'(t) = - \int_0^{L(t)} \langle \frac{\partial \mathcal{C}}{\partial t}, \kappa \mathcal{N} \rangle dv$ . The goal now is to compute from this derivation, the flow minimizing  $L(t)$  as

fast as possible, in an Euclidean invariant way. For this, we need to define an Euclidean invariant metric. In the standard way, we can define a norm  $\| \cdot \|_{euc}$  on the (Fréchet) space of twice-differentiable closed curves in the plane. Indeed, the (curve) norm is given by the length

$$\| \mathcal{C} \|_{euc} := \int_0^1 \| \mathcal{C}_p \| dp = \int_0^1 \langle \mathcal{C}_p, \mathcal{C}_p \rangle^{1/2} dp = \int_0^L dv = L,$$

of the curve  $\mathcal{C}$ . Thus the direction in which  $L(t)$  is decreasing most rapidly is when  $\mathcal{C}$  satisfies the gradient flow  $\mathcal{C}_t = \kappa \mathcal{N}$ , as previously pointed out [8, 9].

We extend now the results above, that is the definition of a curve norm, for the affine group. We use in analogy to the case above the minimization of affine length and area. Since affine geometry is defined only for convex curves [3], we will initially have to restrict ourselves to the (Fréchet) space of differentiable convex closed curves in the plane. Being  $L_{aff} := \oint ds$  the *affine length* [3], we define

$$\| \mathcal{C} \|_{aff} := \int_0^1 \| \mathcal{C}(p) \|_a dp = \int_0^{L_{aff}} \| \mathcal{C}(s) \|_a ds,$$

where  $\| \mathcal{C}(p) \|_a := [\mathcal{C}(p) \times \mathcal{C}_p(p)]$ . Observe that  $\| \mathcal{C}_s \|_a = [\mathcal{C}_s, \mathcal{C}_{ss}] = 1$ ,  $\| \mathcal{C}_{ss} \|_a = [\mathcal{C}_{ss}, \mathcal{C}_{sss}] = \mu$ , where  $\mu$  is the *affine curvature*. This makes the affine norm  $\| \cdot \|_{aff}$  consistent with the properties of the Euclidean norm on curves relative to the Euclidean arc-length  $dv$  ( $\| \mathcal{C}_v \| = 1$ ,  $\| \mathcal{C}_{vv} \| = \kappa$ ).

Note that the area enclosed by  $\mathcal{C}$  is just

$$A = \frac{1}{2} \int_0^1 \| \mathcal{C}(p) \|_a dp = \frac{1}{2} \int_0^1 [\mathcal{C}, \mathcal{C}_p] dp = \frac{1}{2} \| \mathcal{C} \|_{aff} . \quad (3.1)$$

A straightforward computation reveals that the first variation of the area is  $A'(t) = - \int_0^{L_{aff}(t)} [\mathcal{C}_t, \mathcal{C}_s] ds$ . Therefore the affine invariant gradient flow which will decrease the area as quickly as possible relative to  $\| \cdot \|_{aff}$  is exactly  $\mathcal{C}_t = \mathcal{C}_{ss}$ , which, modulo tangential terms, is equivalent to  $\mathcal{C}_t = \kappa^{1/3} \mathcal{N}$  [19], precisely the affine invariant heat equation studied in [1, 19, 20]. Note that based on the Euclidean norm, the flow minimizing the area is  $\mathcal{C}_t = \mathcal{N}$ . Both  $\mathcal{N}$  and  $\mathcal{C}_{ss}$  are normal vectors, each one in its corresponding group. Computations similar to the above show that the affine invariant flow minimizing the affine length is given by  $\mathcal{C}_t = \mu \mathcal{C}_{ss}$  [19].

#### 4. Affine invariant active contours

The goal now is to extend the works in [5, 11] to affine invariant detection. The developments in [5, 11] are strongly based on the original energy-based snakes introduced by Terzopoulos *at al.* [10, 22] as well as the curve evolution

ones in [4, 13]. We refer the interested readers to the mentioned papers for details and relations between the models.

It is important to note that after affine edges are computed locally based on the scale-space or affine gradient derived in [16],<sup>1</sup> affine invariant fitting can be performed (see [23] and references therein). In this work, the affine invariant integration is done by means of active contours.

#### 4.1 Deformable models based on curve shortening

Assume in the 2D case that the deforming curve  $\mathcal{C}$  is given as a level-set of a function  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Then, we can represent the deformation of  $\mathcal{C}$  via the deformation of  $u$ . The 2D segmentation is obtained via the evolution equation [4, 13] ( $u(0, x) = u_0(x)$ )

$$\frac{\partial u}{\partial t} = \phi \|\nabla u\| \operatorname{div} \left( \frac{\nabla u}{\|\nabla u\|} \right) + \nu \phi \|\nabla u\| \quad (t, x) \in [0, \infty] \times \mathbb{R}^2 \quad (4.1)$$

where the stopping term typically has the form  $\phi = (1 + \|\nabla \hat{\Phi}\|^m)^{-1}$ ,  $m = 1$  or 2, and  $\hat{\Phi}$  is a regularized version of the original image  $\Phi$ . Using the fact that  $\operatorname{div} \left( \frac{\nabla u}{\|\nabla u\|} \right) = \kappa$ , where  $\kappa$  is the Euclidean curvature of the level-sets  $\mathcal{C}$  of  $u$ , equation (4.1) can be written in the form  $u_t = \phi \cdot (\nu + \kappa) \|\nabla u\|$ . The flow  $u_t = (\nu + \kappa) \|\nabla u\|$ , means that the the (smooth) level-set  $\mathcal{C}$  of  $u$  we are considering is evolving according to  $\mathcal{C}_t = (\nu + \kappa)\mathcal{N}$ , where  $\mathcal{N}$  is the inward normal to the curve. This equation was first proposed in [17], where extensive numerical research on it was performed. It was introduced in computer vision in [12], where deep research on its importance for shape analysis is presented.

The motion  $\mathcal{C}_t = \kappa\mathcal{N}$ , is the *Euclidean heat flow* presented before, very well know for its excellent geometric smoothing properties [8, 9]. The constant velocity  $\nu\mathcal{N}$ , acts as the balloon force in [6] and is related to classical mathematical morphology. If  $\nu > 0$ , this velocity pushes the curve inwards and it is crucial in the model in order to allow convex initial curves to become non-convex. The external force is given by  $\phi$ , which is supposed to prevent the propagating curve from penetrating into the objects in the image.

This curve evolution model given by (4.1) automatically handles different topologies, allowing the detection any number of objects in the image, without knowing their exact number. This is achieved with the help of the numerical algorithm developed by Osher and Sethian [17].

#### 4.2 Euclidean geodesic active contours

We present now the geodesic active contours derived in [5, 11]. These models are based on the models in [4, 10, 13, 22], as well as the concepts of shortening and gradient flows described in Section 3. In [5], the model is

<sup>1</sup> A local computation of edges is one of the ingredients of active contours schemes.

derived from the principle of least action in physics, showing the mathematical relation between energy and curve evolution based snakes. In [11], the model is derived immediately from curve shortening, and is compared to similar flows in continuum mechanics. One of the basic ideas is to change the ordinary Euclidean arc-length function  $dv = \| \mathcal{C}_p \| dp$  along a curve  $\mathcal{C}(p)$  by multiplying by a conformal factor  $\phi(x, y) > 0$ , which is assumed to be a positive, differentiable function. The resulting *conformal Euclidean metric* on  $\mathbb{R}^2$  is given by  $\phi dx dy$ , and its associated arc length element is  $dv_\phi = \phi dv = \phi \| \mathcal{C}_p \| dp$ . As in ordinary curve shortening, we want to compute the corresponding gradient flow for the modified length functional  $L_\phi(t) := \int_0^L \phi dv = \int_0^1 \| \mathcal{C}_p \| \phi dp$ . Taking the derivative and integrating by parts, we find that  $-L'_\phi(t) = \int_0^{L_\phi(t)} \langle \mathcal{C}_t, \phi \kappa \mathcal{N} - (\nabla \phi \cdot \mathcal{N}) \mathcal{N} \rangle dv$  [5, 11], which based on the Euclidean metric in Section 3, means that the (gradient) direction in which the  $L_\phi$  perimeter is shrinking as fast as possible is given by  $\frac{\partial \mathcal{C}}{\partial t} = \phi \kappa \mathcal{N} - (\nabla \phi \cdot \mathcal{N}) \mathcal{N}$ . As long as the flow remains regular, we will get convergence to a closed geodesic in the plane relative to the conformal Euclidean metric  $\phi dx dy$ . Regularity may be deduced from the classical curve shortening case.

As in [4, 13], we may add an inflationary constant of the form  $\nu \phi \mathcal{N}$ , and embed the flow as a level-set. In the context of image processing, we take  $\phi$  to be a stopping term depending on the image. The new gradient term  $\nabla \phi$  directs the curve towards the boundary of the objects, increasing attraction to them. Existence, uniqueness and stability results for the gradient active contour model above were studied in [5, 11]. See [5, 11] for details, examples, and relations with other active contours schemes.

### 4.3 Affine invariant geodesic active contours

We can now formulate the functionals that will be used to define the affine invariant snakes. Assume that  $\phi_{aff} = \phi(w_{aff})$  is an affine invariant stopping term, based on the affine invariant edge detectors in [16], in analogy with the Euclidean case developed in previous Section. Therefore,  $\phi_{aff}$  behaves as the weight  $\phi$  in  $L_\phi$ , being now affine invariant. As in the Euclidean case, we regard  $\phi_{aff}$  as an affine invariant conformal factor, and replace the affine arc length element  $ds$  by a conformal counterpart  $ds_{\phi_{aff}} = \phi_{aff} ds$  to obtain the first possible functional for the affine active contours

$$L_{\phi_{aff}} := \int_0^{L_{aff}(t)} \phi_{aff} ds. \tag{4.2}$$

The obvious next step is to compute the gradient flow corresponding to  $L_{\phi_{aff}}$  in order to produce the affine invariant model. <sup>2</sup> Unfortunately, as we will

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<sup>2</sup> Using the connection  $ds = \kappa^{1/3} dv$  [19],  $L_{\phi_{aff}} = \int_0^{L(t)} \phi_{aff} \kappa^{1/3} dv$ .

see, this will lead to an impractically complicated geometric contour model which involves four spatial derivatives.

The snake model which we will use comes from another (special) affine invariant, namely *area*, cf. (3.1). Following the results in Section 3, we define the conformal area functional to be

$$A_{\phi_{\text{aff}}} := \int_0^1 [\mathcal{C}, \mathcal{C}_p] \phi_{\text{aff}} dp = \int_0^{L_{\text{aff}}(t)} [\mathcal{C}, \mathcal{C}_s] \phi_{\text{aff}} ds.$$

Using the definition  $Y^\perp := (-y_2, y_1)^T$  where  $(y_1, y_2)^T \in \mathbb{R}^2$ , we obtain

**Lemma 4.1.**

$$\frac{dL_{\phi_{\text{aff}}}(t)}{dt} = - \int_0^{L_{\text{aff}}(t)} [\mathcal{C}_t, (\nabla \phi_{\text{aff}})^\perp] ds + \int_0^{L_{\text{aff}}(t)} \phi_{\text{aff}} \mu [\mathcal{C}_t, \mathcal{C}_s] ds. \quad (4.3)$$

$$\frac{dA_{\phi_{\text{aff}}}(t)}{dt} = - \int_0^{L_{\text{aff}}(t)} [\mathcal{C}_t, (\phi_{\text{aff}} \mathcal{C}_s + \frac{1}{2} [\mathcal{C}, (\nabla \phi)^\perp \mathcal{C}_s])] ds. \quad (4.4)$$

The affine invariance of the resulting variational derivatives follows from a general result governing invariant variational problems having volume preserving symmetry groups proved in [15].

We now consider the corresponding gradient flows computed with respect to  $\|\cdot\|_{\text{aff}}$ . First, the flow corresponding to the functional  $L_{\phi_{\text{aff}}}$  is

$$\mathcal{C}_t = \{(\nabla \phi_{\text{aff}})^\perp + \phi_{\text{aff}} \mu \mathcal{C}_s\}_s = ((\nabla \phi_{\text{aff}})^\perp)_s + (\phi_{\text{aff}} \mu)_s \mathcal{C}_s + \phi_{\text{aff}} \mu \mathcal{C}_{ss}.$$

As before, we ignore the tangential components, which do not affect the geometry of the evolving curve, and so obtain the following possible model for geometric affine invariant active contours:

$$\mathcal{C}_t = \phi_{\text{aff}} \mu \kappa^{1/3} \mathcal{N} + \langle ((\nabla \phi_{\text{aff}})^\perp)_s, \mathcal{N} \rangle \mathcal{N}. \quad (4.5)$$

The geometric interpretation of the affine gradient flow (4.5) minimizing  $L_{\phi_{\text{aff}}}$  is analogue to the one of the corresponding Euclidean geodesic active contours [16]. This flow involves  $\mu$  which makes it difficult to implement.

The gradient flow coming from the first variation of the modified area functional on the other hand is much simpler:

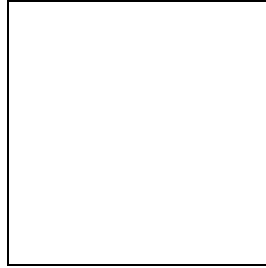
$$\mathcal{C}_t = (\phi_{\text{aff}} \mathcal{C}_s + \frac{1}{2} [\mathcal{C}, (\nabla \phi_{\text{aff}})^\perp] \mathcal{C}_s)_s. \quad (4.6)$$

Ignoring tangential terms, this flow is equivalent to

$$\mathcal{C}_t = \phi_{\text{aff}} \kappa^{1/3} \mathcal{N} + \frac{1}{2} \langle \mathcal{C}, \nabla \phi_{\text{aff}} \rangle \kappa^{1/3} \mathcal{N}. \quad (4.7)$$

Notice that both models (4.5) and (4.7) were derived for *convex curves*, even though the flow (4.7) makes sense in the non-convex case. Formal results regarding existence of (4.7) can be derived following [1, 4, 5, 11].

The figure below illustrates simulations of these active contour models (the implementation is as in [5, 11, 13], based on the level-sets formulation [17]).



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