Classification and Uniqueness of Invariant Geometric Flows [1]

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Abstract – In this note we classify geometric flows invariant to subgroups of the projective group. We proof that the geometric heat flow is the simplest of all possible flows. These results are based on the theory of differential invariants and symmetry groups.

Classification et Unicité des Flux Géométriques Invariants

Résumé – Dans cet article on classifie des flux géométriques, qui sont invariants par des sous-groupes du groupe projectif. On prouve que l'équation de la chaleur géométrique est la plus simple possible. Les résultats se basent sur la théorie des invariants differentiels et des groupes de symmétrie.

Version francaise abrégée

Etant donné $\mathcal{C}(p,t) : S^1 \times [0,\tau) \to \mathbf{R}^2$, une courbe plane simple, et r, la longueur d'arc d'un groupe G, le flux de la chaleur géométrique invariant par le groupe G est donné par l'équation

$$\frac{\partial \mathcal{C}}{\partial t} = \frac{\partial^2 \mathcal{C}}{\partial r^2}. \tag{1}$$

Un flux géométrique plus général est obtenu en incorporant la courbure χ du groupe

$$\begin{aligned} \frac{\partial \mathcal{C}(p,t)}{\partial t} &= \Psi(\chi, \frac{\partial \chi}{\partial r}, \ldots, \frac{\partial^n \chi}{\partial r^n}) \frac{\partial^2 \mathcal{C}(p,t)}{\partial r^2}, \\ \mathcal{C}(p,0) &= \mathcal{C}_0(p). \end{aligned} \tag{2}$$

Pour ces flux on prouvera les résultats suivants:

Lemme 1 – *Localement on peut représenter la solution de (1) comme $y = u(x,t)$, et l'évolution est donnée par*

$$\frac{\partial u}{\partial t} = \frac{1}{g^2}\frac{\partial^2 u}{\partial x^2},$$

quand $g = dr/dx$.

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Théoreme 1 – *Pour tout sous-groupe du group projectif on a:*

1 Tout invariant différentiel est une fonction

$$I\left(\chi, \frac{d\chi}{dr}, \frac{d^2\chi}{dr^2}, \ldots, \frac{d^n\chi}{dr^n}\right).$$

2 Toute évolution invariante est donnée par

$$\frac{\partial u}{\partial t} = \frac{1}{g^2}\frac{\partial^2 u}{\partial x^2}\, I. \tag{3}$$

Corollaire 1 – *Si G est un des groupes suivants (voir remarque sur le groupe Euclidien dans la version anglaise): affine, affine spécial, de similitudes et projectif, alors le flux suivant est le seul invariant d'ordre minimal:*

$$\frac{\partial u}{\partial t} = \frac{c}{g^2}\frac{\partial^2 u}{\partial x^2},$$

où c est une constante.

La première partie du Théorème 1 ne nécessite pas que G soit un sous-groupe de $\mathrm{SL}(\mathbf{R}, 3)$. Cette condition est nécessaire pour la deuxième partie (voir [10] pour extension). La classification peut être construite pour d'autres groupes, mais les relations seront plus compliquées.

L'unicité de l'équation de la chaleur géométrique pour le groupe Euclidien et le groupe affine, est démontrée dans [1] avec une autre méthodologie et sous certaines conditions. Le résultat présenté ici est plus général, parce qu'il n'implique pas une analyse indépendante pour chaque sous-groupe (au contraire du résultat en [1]). On considère aussi que l'interprétation géométrique présenteé ici est claire, intuitive, et décrit naturellement des flux invariants en fonction des invariants primaires de la géométrie différentielle: la longueur d'arc et la courbure.

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1 Introduction

Let $\mathcal{C}(p,t) : S^1 \times [0,\tau) \to \mathbf{R}^2$ ($\mathcal{C} = [x,y]^T$) be a family of simple planar curves, where p parametrizes the curve and t the family (p and t are independent). Assume that we want to formulate an intrinsic *geometric heat flow* for plane curves which is invariant under a certain transformation group G, that is, admits G-invariant solutions. These type of flows replace the classical heat flow, which is equivalent to Gaussian smoothing, and frequently used in image analysis. Let r denote the group arc-length, i.e., the simplest invariant parametrization of the group [8, 9]. Then, the *invariant geometric heat flow* is given by [12]

$$\begin{aligned} \frac{\partial \mathcal{C}(p,t)}{\partial t} &= \frac{\partial^2 \mathcal{C}(p,t)}{\partial r^2}, \\ \mathcal{C}(p,0) &= \mathcal{C}_0(p). \end{aligned} \tag{4}$$

If G acts linearly, it is easy to see that since dr is an invariant of the group, so is $\mathcal{C}_{rr}$. $\mathcal{C}_{rr}$ is called the *group normal*. For nonlinear actions, the flow (4) is still G-invariant (if $\mathcal{C}$ is a solution, so is $G(\mathcal{C})$), since $\frac{\partial}{\partial r}$ is the unique *invariant derivative* [8, 9]. For short and long term existence of the these flows for different groups, see [2, 3, 4, 5, 9, 11, 12].

The flow given by (4) is non-linear, since the group arc-length r is a time-dependent parametrization. This flow gives the invariant geometric heat-type flow of the group, and provides the invariant direction of the deformation. For subgroups of the full projective group SL($\mathbf{R}$, 3), we show in Theorem 1 below that the most general invariant evolutions are obtained if the group curvature χ, i.e., the simplest non-trivial differential invariant of the group, and its derivatives (with respect to arc-length) are incorporated into the flow:

$$\begin{aligned} \frac{\partial \mathcal{C}(p,t)}{\partial t} &= \Psi(\chi, \frac{\partial \chi}{\partial r}, \ldots, \frac{\partial^n \chi}{\partial r^n}) \frac{\partial^2 \mathcal{C}(p,t)}{\partial r^2}, \\ \mathcal{C}(p,0) &= \mathcal{C}_0(p), \end{aligned} \tag{5}$$

where $\Psi(\cdot)$ is a given function. (We discuss the existence of possible solutions of (5) in [12].) Since the group arc-length and group curvature are the basic differential invariants of the group transformations, it is natural to formulate (5) as the most general geometric invariant flow.

2 Uniqueness of Invariant Heat Flows

In this section, we formulate a result, which elucidates in what sense our invariant heat-type equations (4) are unique. We use here the action of the projective group SL($\mathbf{R}$, 3) on $\mathbf{R}^2$. The proofs are based on the theory of Lie groups, prolongations, and symmetry groups. See [7, 8, 9] for this corresponding background. The proofs of the results below may be found in [9]. We will first note that locally, we may express a solution of (4) as the graph of $y = u(x,t)$.

Lemma 1 *Locally, the evolution (4) is equivalent to*

$$\frac{\partial u}{\partial t} = \frac{1}{g^2} \frac{\partial^2 u}{\partial x^2},$$

where g is the G–invariant metric ($g = dr/dx$).

We can now state the following fundamental result from [9]:

Theorem 1 *Let G be a subgroup of the projective group SL($\mathbf{R}$, 3). Let $dr = gdp$ denote the G-invariant arc-length and χ the G-invariant curvature. Then*

1 . *Every differential invariant of G is a function*

$$I\left(\chi, \frac{d\chi}{dr}, \frac{d^2\chi}{dr^2}, \ldots, \frac{d^n\chi}{dr^n}\right)$$

of χ and its derivatives with respect to arc length.

2 . *Every G-invariant evolution equation has the form*

$$\frac{\partial u}{\partial t} = \frac{1}{g^2} \frac{\partial^2 u}{\partial x^2} I, \tag{6}$$

where I is a differential invariant for G. That means, for the given G, all the evolution (curve deformation) equations for which if $\mathcal{C}$ is a solution so is $G(\mathcal{C})$, have the form above.

We are particularly interested in the following subgroups of the full projective group: Euclidean, similarity, special affine, affine, full projective.

Corollary 1 *Let G denote the special affine, full affine, similarity, or full projective group (see remark below for the Euclidean group). Then there is, up to a constant factor, a unique G-invariant evolution equation of lowest order, namely*

$$\frac{\partial u}{\partial t} = \frac{c}{g^2}\frac{\partial^2 u}{\partial x^2},$$

where c is a constant.

Note that the Corollary follows from the fact that, for the listed subgroups, the invariant arc length r depends on lower order derivatives of u than the invariant curvature χ. (This fact holds for most (but *not* all) subgroups of the projective group; one exception is the group consisting of translations in x, u, and scalings $(x,u) \mapsto (\lambda x, \lambda u)$.) For the Euclidean group, it is interesting to note that the simplest nontrivial flow is given by (constant motion)

$$u_t = c\sqrt{1+u_x^2}, \quad c \text{ a constant.}$$

(Here $g = \sqrt{1+u_x^2}$.) In this case the curvature (the ordinary planar curvature κ) has order 2. This equation is obtained for the invariant function $I = 1/\kappa$. The Euclidean geometric heat equation is indeed given by the flow in the Corollary. The orders for the different groups are as follows:

Group	*Arc Length*	*Curvature*
Euclidean	1	2
Similarity	2	3
Special Affine	2	4
Affine	4	5
Projective	5	7

The explicit formulas are given in the following table:

Group	*Arc Length*	*Curvature*
Euclidean	$\sqrt{1+u_x^2}dx$	$\dfrac{u_{xx}}{(1+u_x^2)^{3/2}}$
Similarity	$\dfrac{u_{xx}dx}{(1+u_x^2)}$	$\dfrac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{u_{xx}^2}$
Special Affine	$(u_{xx})^{1/3}dx$	$\dfrac{P_4}{(u_{xx})^{8/3}}$
Affine	$\dfrac{\sqrt{P_4}}{u_{xx}}\,dx$	$\dfrac{P_5}{(P_4)^{3/2}}$
Projective	$\dfrac{(P_5)^{1/3}}{u_{xx}}\,dx$	$\dfrac{P_7}{(P_5)^{8/3}}$

Here

$$\begin{aligned}
P_4 &= 3u_{xx}u_{xxxx} - 5u_{xxx}^2,\\
P_5 &= 9u_{xx}^2 u_{xxxxx} - 45u_{xx}u_{xxx}u_{xxxx} + 40u_{xxx}^3,\\
P_7 &= \frac{1}{3}u_{xx}^2[6P_5 D_x^2 P_5 - 7(D_x P_5)^2] + 2u_{xx}u_{xxx}P_5 D_x P_5 - (9u_{xx}u_{xxxx} - 7u_{xxx}^2)P_5^2.
\end{aligned}$$

Part 1 of the Theorem 1 (suitably interpreted) does not require G to be a subgroup of the projective group; however for part 2 and the corollary this is essential. One can, of course, classify the differential invariants, invariant arc-lengths, invariant evolution equations, etc., for any group of transformations in the plane, but the interconnections are more complicated. See Lie [6] and Olver [8] for the details of the complete classification of all groups in the plane and their differential invariants. See [10] for extensions of invariant flows to other groups and dimensions.

The uniqueness of the Euclidean and affine heat flows (see [11, 12]), was also proven in [1], using a completely different approach, and based on a number of requirements, whereas we only need to impose the conditions of simplicity and group invariance. For example, according to Corollary 1, the unique affine heat flow is obtained by only requiring that it be "the simplest flow" which admits the special affine group as a symmetry group; all the other requirements in [1] can then be proven as properties of the flow — see [11]. Moreover, in contrast with our results, those in [1] were proven independently for each group; when considering a new group, a new analysis must be carried out. Our results in Theorem 1 and Corollary 1 are general, can be applied to any sub-group. (See [10] for extensions to other groups and dimensions.) With the geometric approach here presented, we believe that our result is particularly clear and intuitive.

Finally, it is worth pointing out that besides the Euclidean and affine heat flows, the flows obtained for the other groups in Corollary 1 do not necessarily smooth the curve. From the analysis in [1], it is clear that for example the projective heat flow violates some scale-space properties. In [9], we also show how this flow can develop singularities in an initially smooth curve. (See also [3] for an analysis of projective invariant flows.) Existence and smoothness properties of more general flows, that is for general functions I, are still under research.

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