

A Gradient Surface Evolution Approach to 3D Segmentation

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September 1995

Abstract

In this note, we apply the very general active contour evolution equations formulated in [30, 31] in order to derive new algorithms for 3D segmentation. The algorithm is based on defining feature-based Riemannian metrics in which the feature of interest may be considered to lie at the bottom of a potential well. The segmentation is then carried out relative to a gradient flow equation which generalizes ordinary mean curvature flow.

Key words: 3D segmentation, active vision, gradient flows, Riemannian metrics, mean curvature flow.

This work was supported in part by grants from the National Science Foundation DMS-8811084, ECS-9122106, by the Air Force Office of Scientific Research F49620-94-1-00S8DEF, by the Army Research Office DAAH04-94-G-0054 and DAAH04-93-G-0332.

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1 Introduction

This purpose of this note is to apply the conformal Euclidean curvature flows derived in [30, 31] in order to derive a new algorithm for segmentation and contour finding in volumetric images. Of course, there are already a number of interesting methods devoted to *snakes* or *active contours* both in two and three dimensions described in the literature. The underlying principle is based upon the utilization of deformable contours which conform to various object shapes and motions. Snakes have been used for edge and curve detection, segmentation, shape modelling, and visual tracking. The recent book by Blake and Yuille [9] contains a collection of papers about deformable contours together with a large list of references to which we refer the interested reader.

In the classical frameworks, one considers energy minimization methods where controlled continuity splines are allowed to move under the influence of external image dependent forces, internal forces, and certain constraints set by the user. See [29, 59, 9, 16]. As is well-known there may be a number of problems associated with this approach such as initializations, existence of multiple minima, and the selection of the elasticity parameters.

The present paper uses the theory of conformal Euclidean curvature flows, whose mathematical theory has been extensively developed in [31]. In the three dimensional case considered here, the idea is to multiply the Euclidean area by an image dependent stopping function tailored to the features (e.g., maximum of the norm of the gradient) which we wish to capture in the segmentation. We then compute the corresponding gradient flow which naturally turns out to be a generalization of the classical mean curvature flow in differential geometry used in minimal surface theory. This gives us new 3D active contour models which efficiently attract the initial snake to the features of interest (which basically lie at the bottom of a *potential well*).

We should add that independently in [12, 13, 54] a very similar approach has been implemented. Moreover, there are important surface evolution approaches in [11, 38] which initially motivated this work. There are also related flows in [58, 61].

Finally, the full justification of the evolution equations used in this paper from a viscosity point of view has been carried out in [31].

2 Remarks on Classical Snakes

In this section, we very briefly sketch the energy based optimization approach to deformable contours as discussed in [29, 59, 9]. Our treatment will of course be very incomplete, and once again we refer the interested reader to the collection of papers in [9], especially [60]. For simplicity, we treat the 2D case.

Let $C(p) = (x(p), y(p))^T$ be a closed contour in R^2 where $0 \leq p \leq 1$. (Note that the superscript T denotes transpose.) We now define an energy functional on the set of such contours (“snakes”), $\mathcal{E}(C)$. Following standard practice, we take $\mathcal{E}(C)$ to be of the form

$$\mathcal{E}(C) = \mathcal{E}_{int}(C) + \mathcal{P}(C),$$

where \mathcal{E}_{int} is the *internal deformation energy* and \mathcal{P} is an external potential energy which depends on the image. (Other external constraint forces may be added.) A common choice for the internal energy is the quadratic functional

$$\mathcal{E}_{int}(C) := \int_0^1 w_1(p) \|C_p\|^2 + w_2(p) \|C_{pp}\|^2 dp,$$

where w_1 and w_2 control the “tension” and “rigidity” of the snake, respectively. (Note that the subscripts denote derivatives with respect to p in the latter expression, and $\|\cdot\|$ denotes the standard Euclidean norm.)

Let $I : R^2 \rightarrow R$ be the given grey-scale image. Then the external potential energy depends on the image $I(x, y)$. It can be defined by

$$\mathcal{P}(C) := \int_0^1 P(C(p)) dp,$$

where $P(x, y)$ is a scalar potential function defined on the image plane. The local minima of P attract the snake. For example, we may choose P to be

$$P(x, y) := c \|\nabla G_\sigma * I(x, y)\|,$$

for a suitably chosen constant c , in which case the snake will be attracted to intensity edges. Here G_σ denotes a Gaussian smoothing filter of standard deviation σ .

One also typically considers dynamic time-varying models in which $C(p)$ becomes a function of time as well; see [60]. In this case, one defines a kinetic

energy and the corresponding Lagrangian (the difference between the kinetic energy and the energy \mathcal{E} defined above). Applying the principle of least action, one derives the corresponding Lagrange equation which one tries to solve numerically employing various approximations.

In the approach to be given below, we will also use an energy method. However, in contrast to more *ad hoc* approaches, we believe that our energy is intrinsic to the given geometry of the problem, as is the correspondent gradient flow. Moreover, as we will see, kinetic energy is not needed for a successful snake algorithm, and indeed potential energy suffices if chosen in the “proper” manner.

3 3-D Active Contour Models

In this section, we will formulate our geometric 3-D contour models based on surface evolution ideas. This model is derived by modifying the Euclidean area by a function which depends on the salient image features which we wish to capture. In order to do this, we will need to set up some notation. (For all the relevant concepts on the differential geometry of surfaces, we refer the reader to [19].)

Let $S : [0, 1] \times [0, 1] \rightarrow R^3$ denote a compact embedded surface with (local) coordinates (u, v) . Let H denote the mean curvature and $\vec{\mathcal{N}}$ the inward unit normal. (Recall that H is the arithmetic mean of the principal curvatures [19].) We set

$$S_u := \frac{\partial S}{\partial u}, \quad S_v := \frac{\partial S}{\partial v}.$$

Then the infinitesimal area on S is given by

$$dS = (\|S_u\|^2 \|S_v\|^2 - \langle S_u, S_v \rangle^2)^{1/2} dudv.$$

Let $\phi : \Omega \rightarrow R$ be a positive differentiable function defined on some open subset of R^3 . The function $\phi(x, y, z)$ will play the role of a “stopping” function. Thus the function $\phi(x, y, z)$ will depend on the given grey-level image. For example, the term $\phi(x, y, z)$ may chosen to be small near a 3D edge, and so acts to stop the evolution when the 3D contour reaches the edge. In this paper, we will choose

$$\phi := \frac{1}{1 + \|\nabla G_\sigma * I\|^2}, \tag{1}$$

where $I = I(x, y, z)$ is the (grey-scale) volumetric image and G_σ is a Gaussian (smoothing) filter.

It is a beautiful classical fact that the gradient flow associated to the area functional for surfaces (i.e., the direction in which area is shrinking most rapidly) is given by

$$\frac{\partial S}{\partial t} = H\vec{\mathcal{N}}. \quad (2)$$

(See [10, 24, 39, 42, 62] and the references therein.) What we propose to do is to replace the Euclidean area by a modified area depending on ϕ namely,

$$dS_\phi := \phi dS.$$

For a family of surfaces (with parameter t), consider the ϕ -area functional

$$A_\phi(t) := \int \int_S dS_\phi.$$

Then a simple integration by parts argument gives that

$$\frac{dA_\phi}{dt} = - \int \int_S \left\langle \frac{\partial S}{\partial t}, \phi H\vec{\mathcal{N}} - \nabla\phi + \text{tangential components} \right\rangle dS.$$

Since the tangential part only affects the parametrization, it may be dropped, leading to the model

$$\frac{\partial S}{\partial t} = \phi H\vec{\mathcal{N}} - \nabla\phi. \quad (3)$$

As in [11, 38], a constant inflation term ν may be added to give the model

$$\frac{\partial S}{\partial t} = \phi(H + \nu)\vec{\mathcal{N}} - \nabla\phi. \quad (4)$$

This inflationary constant may be taken to be either positive (inward evolution) or negative in which case it would have an outward or expanding effect. For sufficiently large ν , this would cause the evolution to act as an expanding “balloon” or “bubble” [16, 58].

The level set version of (3) [43, 44, 50, 51, 52] is given in terms of the evolving level set function $\Psi(x, y, z, t)$ by

$$\frac{\partial \Psi}{\partial t} = \phi \|\nabla \Psi\| \operatorname{div} \left(\frac{\nabla \Psi}{\|\nabla \Psi\|} \right) + \nabla \phi \cdot \nabla \Psi. \quad (5)$$

Here again one may add a constant inflation term to the mean curvature to derive the level set version of (4)

$$\Psi_t = \phi \|\nabla \Psi\| \left(\operatorname{div} \left(\frac{\nabla \Psi}{\|\nabla \Psi\|} \right) + \nu \right) + \nabla \phi \cdot \nabla \Psi. \quad (6)$$

(As usual we take Ψ to be negative in the interior and positive in the exterior of the zero level set.)

The 2D versions of (4) and (6) were implemented in [30, 31, 64]. Despite their apparent similarities, there are also some important differences between the 2D and 3D models discussed here. Indeed, the geometric heat equation will shrink a simple closed curve to a round point without developing singularities, even if the initial curve is *nonconvex* [26]. The geometric model for active 2D contours from [30, 31] is based on this flow. For surfaces, it is well-known that singularities may develop in the mean curvature flow (2) non-convex smooth surfaces [25]. (The classical example is the dumbbell.) We should note however that the mean curvature flow does indeed shrink smooth compact convex surfaces to round “spherical” points; see [28].

We should add that because of these problems, several researchers have proposed replacing mean curvature flow by flows which depend on the Gaussian curvature κ . Indeed, define

$$\kappa_+ := \max\{\kappa, 0\}.$$

Then Caselles and Sbert [14] have proven that the *affine invariant surface flow*

$$\frac{\partial S}{\partial t} = \operatorname{sign}(H) \kappa_+^{1/4} \vec{\mathcal{N}} \quad (7)$$

will (smoothly) shrink rotationally symmetric compact surfaces to ellipsoidal shaped points. (This has been proven in [7] in the convex case. See also [3, 6].) Thus one could replace the mean curvature part by $\operatorname{sign}(H) \kappa_+^{1/4}$ in (6). Another possibility would be to use $\kappa_+^{1/2}$ as has been proposed in [40]. See also [58]. (Note that Chow [15] has demonstrated that convex surfaces flowing under $\kappa^{1/2}$ shrink to spherical points.) All these possible evolutions for 3D contours are now being explored and will be reported in a future publication.

4 Numerical Implementations

We will now describe a numerical experiment to illustrate our methods. One of the most successful implementations of curvature flow equations is that based on the Osher-Sethian level set formulations of the evolutions in [43, 44, 50, 51, 52, 38]. This formulation is global, and so effectively increases the problem dimension by one. (More local implementations may be found in the recent work [1].) Since we have been working with volumetric data, we have found it advantageous to use methods that do not increase the dimensionality but still keep the advantages of the Osher-Sethian level set algorithm (i.e., the ability to handle singularities and topological changes such as breaking, merging, and extinction). This means that we have not implemented the level set version (6) of the generalized mean curvature flow (4), but instead have chosen another numerical strategy.

The numerics we have implemented are reported in [63]. One possible method is based on combining a singular perturbed reaction-diffusion equation (an Allen-Cahn type equation) with a dynamic mesh algorithm as in [41]. The modified mean surface evolution equation can then be directly implemented without paying the price of going up in dimension which may be very expensive for volumetric data sets.

The simulations which we give below make use of such a methodology and will be reported in full in [63]. All of our runs were done on an Silicon Graphics single processor Indigo machine. The segmentation was accomplished in 60 iterations which ran for less than 10 seconds on the computer. The volumetric image itself was $128 \times 128 \times 60$. Figure 1 shows a slice of the original MRI brain image together with an initial spherical bubble which was placed in the ventricles, and Figure 2 indicates the resulting segmentation. No noise filtering was performed on the image.

5 Conclusions

In this note, we have considered a natural differential geometric approach based on image-dependent Riemannian metrics and the associated gradient flows for active 3D contour models. This approach yields evolution equations which are the natural generalization of the classical mean curvature flow to minimal surfaces defined relative to a conformal Euclidean metric.

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