Math 3592H Honors Math I
Midterm exam 2, Thursday November 10, 2016

Instructions:
50 minutes, closed book and notes, no electronic devices.
There are four problems, worth a total of 100 points.

1. (30 points; 10 points each part)

Let A be a 3×5 matrix.

(i) Prove or disprove: there are no vectors $\overline{\mathbf{b}}$ in $\mathbb{R}^3$ for which $A\overline{\mathbf{x}} = \overline{\mathbf{b}}$ has exactly one solution $\overline{\mathbf{x}}$ in $\mathbb{R}^5$.

True. Row-reducing $[A|\overline{b}] \rightsquigarrow [\tilde{A}|\tilde{b}]$ in echelon form will either have a pivotal 1 in the last ($\tilde{b}$) column, so no solutions, or at least one non-pivotal column in $\tilde{A}$ (there are 5 columns, and ≤ 3 pivotal columns since only 3 rows), and ∞ many solutions.

(ii) Now assume A can be row-reduced to $\tilde{A} = \begin{bmatrix} 0 & 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Write down a basis for the subspace $V = \{\overline{\mathbf{x}} \in \mathbb{R}^5 : A\overline{\mathbf{x}} = \overline{\mathbf{0}}\}$.

$$A\overline{x} = \overline{0} \iff \tilde{A}\overline{x} = \overline{0} \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_3 + x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow V$ has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, since the above expression is unique - look at the x_1, x_3, x_4 coordinates.

(iii) Write down a matrix E having the following property:

if $A = \begin{bmatrix} \bar{\mathbf{r}}_1^\top \\ \bar{\mathbf{r}}_2^\top \\ \bar{\mathbf{r}}_3^\top \end{bmatrix}$ with $\bar{\mathbf{r}}_i$ in $\mathbb{R}^5$, then $EA = \begin{bmatrix} \bar{\mathbf{r}}_1^\top \\ \bar{\mathbf{r}}_2^\top \\ \bar{\mathbf{r}}_3^\top - 6\bar{\mathbf{r}}_1^\top \end{bmatrix}$

This is one of our elementary matrices : $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix}$

2. (20 points total) Prove or disprove: If $\bar{\mathbf{f}}, \bar{\mathbf{g}} : \mathbb{R}^4 \to \mathbb{R}^4$ are both differentiable everywhere, and $(\bar{\mathbf{f}} \circ \bar{\mathbf{g}}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}$ for all $\bar{\mathbf{x}}$ in $\mathbb{R}^4$, then the Jacobian matrix $[J\bar{\mathbf{f}}(\bar{\mathbf{a}})]$ is invertible for every[1] $\bar{\mathbf{a}}$ in $\mathrm{img}(\bar{\mathbf{g}})$.

True: As usual, chain rule for derivatives at $\bar{x} = \bar{b}$ gives

$$[J\bar{f}(\bar{g}(\bar{b}))][J\bar{g}(\bar{b})] = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}$$

Jacobian for $h\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}$ (at $\bar{x} = \bar{b}$)

and assuming $\bar{a} \in \mathrm{img}(\bar{g})$, $\exists\, \bar{b}_0$ with $g(\bar{b}_0) = \bar{a}$, so we get

$$[J\bar{f}(\bar{a})][J\bar{g}(\bar{b})] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and $\underbrace{[J\bar{f}(\bar{a})][J\bar{g}(\bar{b})]}[J\bar{f}(\bar{a})][J\bar{g}(\bar{b})] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$

This must be $[J\bar{f}(\bar{a})]^{-1}$, since $J\bar{f}(\bar{a})$ is 4×4 square.

[1] The exam had "for every $\bar{\mathbf{a}}$ in $\mathbb{R}^4$", which is not the assumption I intended!

3. (20 points total; 10 points each part) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & \alpha \end{bmatrix}$.

(i) Assuming that $A\overline{\mathbf{x}} = \overline{\mathbf{0}}$ has infinitely many solutions, what is α?

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & \alpha \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightsquigarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 4 & \alpha & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & 4 & \alpha & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & \alpha-3 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \alpha-4 & 0 \end{array}\right]$$

need $\alpha - 4 = 0$ i.e. $\alpha = 4$
to avoid having 3 pivotal 1's
and unique solution

(ii) Assuming that α is chosen as in the answer to part (i), write down at least one explicit $\overline{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in $\mathbb{R}^3$ so that $A\overline{\mathbf{x}} = \overline{\mathbf{b}}$ has no solutions.

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & b_1 \\ 1 & 1 & 1 & b_2 \\ 3 & 4 & 4 & b_3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_2 \\ 0 & 1 & 1 & b_1 \\ 3 & 4 & 4 & b_3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_2 \\ 0 & 1 & 1 & b_1 \\ 0 & 1 & 1 & b_3 - 3b_2 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_2 \\ 0 & 1 & 1 & b_1 \\ 0 & 0 & 0 & b_3 - 3b_2 - b_1 \end{array}\right]$$

has no solutions whenever $b_3 - 3b_2 - b_1 \neq 0$

e.g. $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

4. (30 points total; 10 points each part) Prove or disprove:

(a) If $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2$ are nonzero, nonparallel vectors in $\mathbb{R}^3$, then $\{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_1 \times \bar{\mathbf{v}}_2\}$ are linearly independent. True.

Note $\bar{v}_1, \bar{v}_2$ nonzero, nonparallel

$\Rightarrow \bar{v}_1 \times \bar{v}_2 \neq \bar{0}$, since $|\bar{v}_1 \times \bar{v}_2|$ = area of parallelogram $\{\bar{0}, \bar{v}_1, \bar{v}_2, \bar{v}_1 + \bar{v}_2\}$

If $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\,\bar{v}_1\times\bar{v}_2 = \bar{0}$, then dotting with $\bar{v}_1\times\bar{v}_2$ gives $c_3\underbrace{|\bar{v}_1\times\bar{v}_2|^2}_{\neq 0} = 0 \Rightarrow \boxed{c_3 = 0}$

and hence $c_1\bar{v}_1 + c_2\bar{v}_2 = \bar{0}$ $\Rightarrow \boxed{c_1 = c_2 = 0}$ since ~~[illegible]~~ $\bar{v}_1, \bar{v}_2$ ~~[illegible]~~ are nonzero, nonparallel (hence lin. indep.)

(b) For any angle θ, the vectors $\bar{\mathbf{v}}_1 = \begin{bmatrix} -\cos(6\theta) \\ -\sin(6\theta) \end{bmatrix}$, $\bar{\mathbf{v}}_2 = \begin{bmatrix} \sin(6\theta) \\ -\cos(6\theta) \end{bmatrix}$ are orthonormal in $\mathbb{R}^2$. True.

Check $|\bar{v}_1|^2 = |\bar{v}_2|^2 = \cos^2(6\theta) + \sin^2(6\theta) = 1$

and $\bar{v}_1 \cdot \bar{v}_2 = -\cos(6\theta)\sin(6\theta) + \sin(6\theta)\cos(6\theta) = 0$.

(b) For any angle θ, the vectors $\bar{\mathbf{v}}_1 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$, $\bar{\mathbf{v}}_2 = \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix}$ are orthonormal in $\mathbb{R}^2$. False.

e.g. for $\theta = 0$, $\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \bar{v}_2$ so $\bar{v}_1 \cdot \bar{v}_2 \neq 0$.