Math 3592H Honors Math I

Instructions: Quiz 2, Thursday Oct. 20, 2016

15 minutes, closed book and notes, no electronic devices.
There are three problems, worth a total of 20 points.

1. (8 points total; 4 points each part)

Consider the function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} xy \\ z^2 \end{pmatrix}$.

(a) Write down the matrix representing $D\mathbf{f}(\mathbf{a})$ at $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Since $\bar{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ has both f_1, f_2 polynomial, it is differentiable everywhere on $\mathbb{R}^3$, with

$$\left[D\bar{f}\begin{pmatrix}1\\1\\1\end{pmatrix}\right] = \underbrace{J\bar{f}\begin{pmatrix}1\\1\\1\end{pmatrix}}_{\text{Jacobian matrix}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix}\right|_{\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}} = \left.\begin{bmatrix} y & x & 0 \\ 0 & 0 & 2z \end{bmatrix}\right|_{\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) Compute the directional derivative of $\mathbf{f}$ at $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in the direction of the unit vector $\mathbf{v} = \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix}$.

Since $\bar{f}$ is differentiable at $\begin{pmatrix}1\\1\\1\end{pmatrix}$, this directional derivative is

$$\left[J\bar{f}\begin{pmatrix}1\\1\\1\end{pmatrix}\right]\bar{v} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 0 \end{bmatrix}$$

2. (8 points total; 4 points each part)
Assume $\mathbf{f} : \mathcal{U} \to \mathbb{R}^{100}$ is defined on an open subset $\mathcal{U}$ of $\mathbb{R}^3$, and differentiable at some point $\mathbf{a} \in \mathcal{U}$.

(a) What are the dimensions of the Jacobian matrix $J\mathbf{f}(\mathbf{a})$?

$J\bar{f}(\bar{a})$ represents a linear transformation $\mathbb{R}^3 \xrightarrow[D\bar{f}(\bar{a})]{} \mathbb{R}^{100}$, so it is $\underbrace{100}_{\text{rows}} \times \underbrace{3}_{\text{columns}}$

(b) On what subset of points $\mathbf{x} \in \mathbb{R}^3$ is the derivative $D\mathbf{f}(\mathbf{a})(\mathbf{x})$ defined?

$D\bar{f}(\bar{a}) : \mathbb{R}^3 \to \mathbb{R}^{100}$, i.e. it is defined on <u>all</u> of $\mathbb{R}^3$

3. (4 points total) Prove or disprove:
The function $\mathbf{f} : \mathrm{Mat}(n,n) \to \mathrm{Mat}(n,n)$ sending $X \in \mathrm{Mat}(n,n)$ to

$$f(X) = I + 6X - 5X^2$$

is differentiable on all of $\mathrm{Mat}(n,n)$, and at $X = A$, its derivative $D\mathbf{f}(A) : \mathrm{Mat}(n,n) \to \mathrm{Mat}(n,n)$ is

$$D\mathbf{f}(A)(H) = 6H - 5(AH + HA).$$

True, since, e.g.

$$\lim_{H \to \bar{0}} \frac{\bar{f}(A+H) - \bar{f}(A) - (6H - 5(AH+HA))}{|H|} =$$

$$\lim_{H \to \bar{0}} \frac{I + 6(A+H) - 5(A+H)^2 - (I + 6A - 5A^2) - (6H - 5(AH+HA))}{|H|} =$$

$$\lim_{H \to \bar{0}} \frac{\cancel{I} + \cancel{6A} + \cancel{6H} - \cancel{5A^2} - \cancel{5(AH+HA)} - 5H^2 - \cancel{I} - \cancel{6A} + \cancel{5A^2} - \cancel{6H} + \cancel{5(AH+HA)}}{|H|} =$$

$$-5 \lim_{H \to 0} \frac{H^2}{|H|} = 0$$

since $\left|\frac{H^2}{|H|}\right| \leq \frac{|H|^2}{|H|} = |H| \to 0$ as $H \to \bar{0}$

(or you could appeal to $\bar{g}(X) = X^2$ having derivative $D\bar{g}(A)(H) = AH + HA$ and using limit laws for sums, scalings, etc)