Proof of Inverse Function Theorem

(following N. Wallach's expansion of M. Spivak's proof from "Calculus on manifolds")
↳ see syllabus page for PDF

Recall we're given $\bar{f}: \underset{\mathbb{R}^n}{\overset{\cap}{U^{open}}} \to \mathbb{R}^n$, $\bar{f} \in C^1(U)$, $\det J\bar{f}(\bar{a}) \neq 0$ for some $\bar{a} \in U$

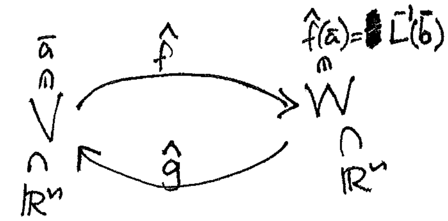
and want to exhibit open sets $\bar{a} \in V (\subset U) \subset \mathbb{R}^n$ and $\bar{g}: W \to V$
$f(\bar{a}) =: \bar{b} \in W \subset \mathbb{R}^n$

such that (i) $\bar{f}: V \to W$ are inverses
$\bar{g}: W \to V$
(ii) $\bar{g}$ is differentiable on W.

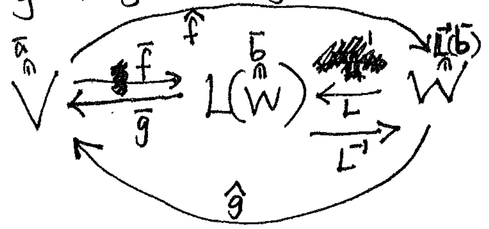
FIRST a reduction to ease computations: we can assume WLOG that $D\bar{f}(\bar{a}) = 1_{\mathbb{R}^n}$ since if $L := J\bar{f}(\bar{a})$ we can replace $\bar{f}$ with the composite $U^{open} \xrightarrow{\bar{f}} \mathbb{R}^n \xrightarrow{L^{-1}} \mathbb{R}^n$ having $D\hat{f}(\bar{a}) = L^{-1} \circ L = 1_{\mathbb{R}^n}$

$\hat{f} = L^{-1} \circ \bar{f}$

If we then find the inverses $\hat{g}$ for $\hat{f}$ with



we can check that $\bar{g} := L^{-1} \circ \hat{g}$ does the job:



SECOND, we can shrink U to a ball $B_\delta(\bar{a})$ of small radius $\delta > 0$ about $\bar{a}$ so as to make these things both happen:

- $\left| \frac{\partial f_i}{\partial x_j}(\bar{x}) - \frac{\partial f_i}{\partial x_j}(\bar{a}) \right| < \frac{1}{2n^2} \quad \forall i,j \quad \forall \bar{x} \in U$ (as $\frac{\partial f_i}{\partial x_j}$ are continuous)
- $\det J\bar{f}(\bar{x}) \neq 0 \quad \forall \bar{x} \in U$, (as $\det J\bar{f}(\bar{x})$ is continuous as a composite

$$\begin{array}{ccccc} U & \longrightarrow & Mat(n,n) & \xrightarrow{\det} & \mathbb{R} \\ \bar{x} & \longmapsto & \left[\frac{\partial f_i}{\partial x_j}\right] & & \\ & & A & \longmapsto & \det A \end{array}$$

so pick δ small enough that $|\det J\bar{f}(\bar{x}) - \det J\bar{f}(\bar{a})| < \frac{|\det J\bar{f}(\bar{a})|}{2}$)
where $\det J\bar{f}(\bar{a}) \neq 0$

The point of that $\frac{1}{2n^2}$ was anticipating a Lipschitz bound we'll use:

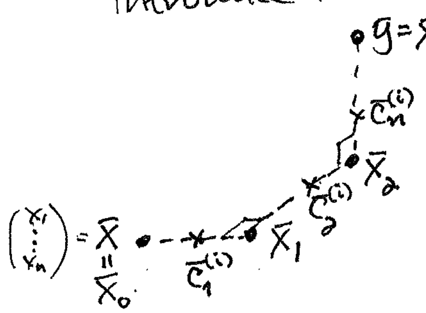
<u>LEMMA</u>: If $\bar{g} : \mathcal{U} \longrightarrow \mathbb{R}$ (open ball) has $\bar{g} \in C^1(\mathcal{U})$ and $\left|\frac{\partial g_i}{\partial x_j}(\bar{x})\right| \leq M \ \forall \bar{x} \in \mathcal{U}$

then $\bar{g}$ satisfies a Lipschitz condition with Lipschitz ~~constant~~ $n^2 M$,

i.e. $|\bar{g}(\bar{y}) - \bar{g}(\bar{x})| \leq n^2 M \, |\bar{y} - \bar{x}| \quad \forall \bar{x}, \bar{y} \in \mathcal{U}$

<u>proof</u>: Just like our proof of THM 1.9.8 $C^1 \Rightarrow$ diff'ble.

Introduce intermediate points $\bar{x}_0, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ inside $\mathcal{U}$, where $\bar{x}_0 = \bar{x}$ and $\bar{x}_n = \bar{y}$



such that $\bar{x}_j - \bar{x}_{j-1} = (y_j - x_j)\bar{e}_j$

and use 1-variable MVT to find $\bar{c}_1^{(i)}, \bar{c}_2^{(i)}, \ldots, \bar{c}_n^{(i)}$ (do this for each $i = 1, \ldots, n$) on the segments between them having

$$f_i(\bar{x}_j) - f_i(\bar{x}_{j-1}) = \frac{\partial f_i}{\partial x_j}(\bar{c}_j^{(i)})(y_j - x_j)$$

and so $f_i(\bar{y}) - f_i(\bar{x}) = \sum_{j=1}^{n} \left(f_i(\bar{x}_j) - f_i(\bar{x}_{j-1})\right) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\bar{c}_j^{(i)})(y_j - x_j)$

and $|\bar{f}(\bar{y}) - \bar{f}(\bar{x})| \leq \sum_{i=1}^{n} |f_i(\bar{y}) - f_i(\bar{x})|$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \underbrace{\left|\frac{\partial f_i}{\partial x_j}(\bar{c}_j^{(i)})\right|}_{\leq M} \underbrace{|y_j - x_j|}_{\leq |\bar{y} - \bar{x}|} \leq n^2 M \cdot |\bar{y} - \bar{x}| \quad \blacksquare$$

This shows that on our <u>shrunken</u> ball $\mathcal{U}$, far away points have far away $\bar{f}$ values:

<u>CLAIM 1</u>: $|\bar{f}(\bar{x}) - \bar{f}(\bar{y})| \geq \frac{1}{2} |\bar{x} - \bar{y}|$ for $\bar{x}, \bar{y} \in \mathcal{U}$

<u>Proof</u>: We'll apply the LEMMA to $\bar{g}(\bar{x}) := \bar{f}(\bar{x}) - \bar{x}$, which has

$$\left|\frac{\partial g_i}{\partial x_j}(\bar{x})\right| = \left|\frac{\partial f_i}{\partial x_j}(\bar{x}) - \delta_{ij}\right| \overset{D\bar{f}(\bar{a}) = I}{=} \left|\frac{\partial f_i}{\partial x_j}(\bar{x}) - \frac{\partial f_i}{\partial x_j}(\bar{a})\right| \leq \frac{1}{2n^2} =: M \quad \forall \bar{x} \in \mathcal{U}.$$

triangle ineq. $\Rightarrow$ $|\bar{v}| - |\bar{w}| \leq |\bar{v} - \bar{w}|$

Now $|\bar{x} - \bar{y}| - |\bar{f}(\bar{x}) - \bar{f}(\bar{y})| \overset{\text{triangle ineq.}}{\leq} |(\bar{f}(\bar{x}) - \bar{x}) - (\bar{f}(\bar{y}) - \bar{y})|$

$$= |\bar{g}(\bar{x}) - \bar{g}(\bar{y})|$$

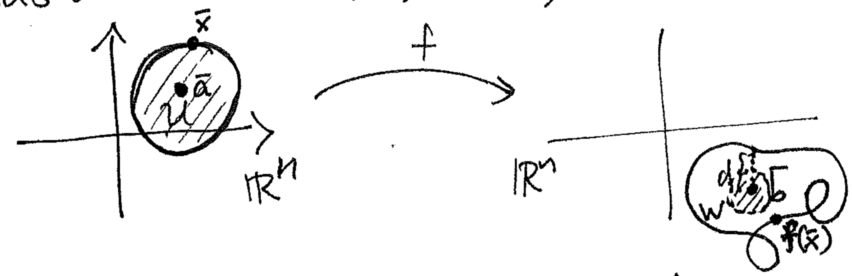
$$\overset{\text{LEMMA}}{\leq} n^2 \cdot \frac{1}{2n^2} |\bar{x} - \bar{y}| = \frac{1}{2} |\bar{x} - \bar{y}| \quad \blacksquare$$

Now shrink $\mathcal{U}$ to an even smaller radius ball $B_\delta(\bar{a})$

so as to make $\dfrac{|\bar{f}(\bar{a}+\bar{h})-\bar{f}(\bar{a})-\bar{h}|}{|\bar{h}|} < 1 \quad \forall |\bar{h}| \leq \delta$ (using $D\bar{f}(\bar{a}) = 1_{\mathbb{R}^n}$ so $D\bar{f}(\bar{a})(\bar{h}) = \bar{h}$)

which then forces $\bar{f}(\bar{a}+\bar{h}) \neq \bar{f}(\bar{a})$, else $\dfrac{|\bar{f}(\bar{a}+\bar{h})-\bar{f}(\bar{a})-\bar{h}|}{|\bar{h}|} = \dfrac{|0-\bar{h}|}{|\bar{h}|} = 1$.

Thus we now have $\bar{f}(\bar{x}) \neq \overset{=\bar{b}}{f(\bar{a})} \quad \forall \bar{x} \in \partial\mathcal{U}$



If we consider [redacted] the continuous function $\partial\mathcal{U} \longrightarrow \mathbb{R}$, $\bar{x} \longmapsto |\bar{f}(\bar{x})-\bar{f}(\bar{a})|$

on the compact set $\partial\mathcal{U}$, it achieves some minimum value $d > 0$,

so we can lower bound $|\bar{f}(\bar{x})-\bar{f}(\bar{a})| \geq d \quad \forall \bar{x} \in \partial\mathcal{U}$.

This finally lets us define $W := B_{d/2}(\bar{b}) = B_{d/2}(\bar{f}(\bar{a}))$
$= \{\bar{y} \in \mathbb{R}^n : |\bar{y}-\bar{f}(\bar{a})| < d/2\}$

By construction $\forall \bar{y} \in W, \bar{x} \in \partial\mathcal{U} \quad |\bar{y}-\bar{f}(\bar{a})| \overset{(*)}{<} |\bar{y}-\bar{f}(\bar{x})|$

CLAIM 2: $\forall \bar{y} \in W \ \exists$ a unique $\bar{x} \in \mathcal{U}$ with $\bar{f}(\bar{x}) = \bar{y}$

proof: In fact, $\bar{x}$ will be any point in the (compact) closed ball $\overline{\mathcal{U}}$ that achieves the minimum value of the continuous function $h: \overline{\mathcal{U}} \longrightarrow \mathbb{R}$

$$\bar{x} \longmapsto h(\bar{x}) = |\bar{y}-\bar{f}(\bar{x})|^2 = \sum_{i=1}^{n}(y_i - f_i(\bar{x}))^2$$

(117) To see this, note that h can't achieve its minimum on $\partial\mathcal{U}$ by the inequality (*), so it achieves it at some $\bar{x} \in \mathcal{U}$ and then one must have $0 = \frac{\partial h}{\partial x_j}(\bar{x}) = \sum_{i=1}^{n} 2(y_i - f_i(\bar{x}))\frac{\partial f_i}{\partial x_j}(\bar{x}) \quad \forall j=1,\dots,n$

$$\Rightarrow \quad \bar{0} = \left[J\bar{f}(\bar{x})\right](\bar{y} - \bar{f}(\bar{x}))$$

$\det J\bar{f}(\bar{x}) \neq 0$ $\forall \bar{x} \in \mathcal{U}$

$$\Rightarrow \quad \bar{0} = \bar{y} - \bar{f}(\bar{x}) \quad , \text{ i.e. } \bar{f}(\bar{x}) = \bar{y}.$$

12/13/2016 Uniqueness of $\bar{x}$ follows because we showed $\forall \bar{x}, \bar{x}' \in \mathcal{U}$ that $|\bar{f}(\bar{x}) - \bar{f}(\bar{x}')| \geq \frac{1}{2}|\bar{x} - \bar{x}'|$, so if $\bar{f}(\bar{x}') = \bar{y} = \bar{f}(\bar{x})$ then $|\bar{x} - \bar{x}'| \leq \frac{1}{2} \cdot 0$ i.e. $\bar{x}' = \bar{x}$. ■

So now if we define $V := \{\bar{x} \in \mathcal{U} : \bar{f}(\bar{x}) \in W\}$

then as maps of sets, we have (from CLAIM 2) $V \underset{\bar{g}}{\overset{\bar{f}}{\rightleftarrows}} W$ are 2-sided inverses

$\bar{f} \circ \bar{g} = 1_W$
$\bar{g} \circ \bar{f} = 1_V$

Also, it is an easy exercise to check that since $\bar{f}$ is continuous and W is open, V will also be open.

CLAIM 1 also shows $\bar{g}$ is continuous, since $\forall \epsilon > 0$, if we choose $\delta = \frac{\epsilon}{2}$ then we find that for $\bar{y}_1, \bar{y}_2 \in W$ with $|\bar{y}_1 - \bar{y}_2| < \delta = \frac{\epsilon}{2}$

the elements $\bar{x}_1 = \bar{g}(\bar{y}_1)$, $\bar{x}_2 = \bar{g}(\bar{y}_2)$ have $\bar{y}_1 = \bar{f}(\bar{x}_1)$, $\bar{y}_2 = \bar{f}(\bar{x}_2)$ so by CLAIM 1,

$$|\bar{f}(\bar{x}_1) - \bar{f}(\bar{x}_2)| \geq \frac{1}{2}|\bar{x}_1 - \bar{x}_2|$$

$$= \frac{1}{2}|\bar{g}(\bar{y}_1) - \bar{g}(\bar{y}_2)|$$

$$\frac{\epsilon}{2} > |\bar{y}_1 - \bar{y}_2| = |\bar{f}(\bar{x}_1) - \bar{f}(\bar{x}_2)|$$

$$\Rightarrow |\bar{g}(\bar{y}_1) - \bar{g}(\bar{y}_2)| < \epsilon.$$

It remains to show $\bar{g}$ is <u>differentiable</u> at every $\bar{y} \in W$.

In fact, we'll check that if $\bar{g}(\bar{y}) = \bar{x} \in V$ (so $\bar{f}(\bar{x}) = \bar{y}$), and if $A := D\bar{f}(\bar{x})$

then $A^{-1} = D\bar{g}(\bar{y})$, as we'd expect from chain rule.