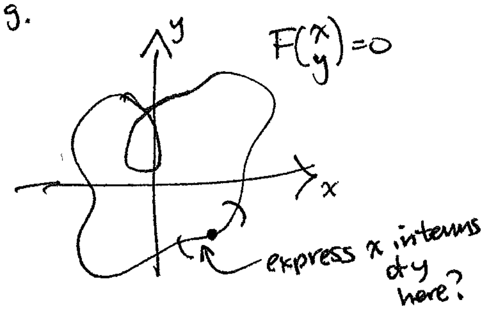
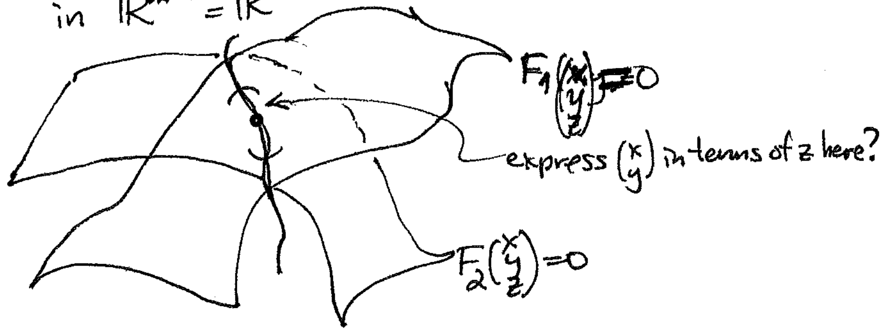
Implicit Function Thm

Suppose we are looking at a solution set in $\mathbb{R}^{n+m}$ of n equations in the $n+m$ unknowns.

e.g. in $\mathbb{R}^{\overset{n}{1}+\overset{m}{1}}=\mathbb{R}^2$



in $\mathbb{R}^{\overset{n}{2}+\overset{m}{1}}=\mathbb{R}^3$



We might expect at <u>most</u> points on the solution set, we can pick a neighborhood and m ("nonpivotal") variables $y_1,\dots,y_m$ that let ~~us~~ locally express/parametrize the n ("pivotal") variables left $x_1,\dots,x_n$ as $\bar{x}=\bar{g}(\bar{y})$

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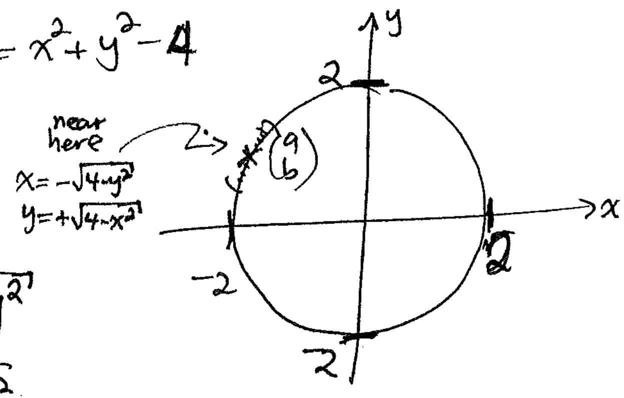
e.g. in $\mathbb{R}^2$, on solution ~~set~~ to $F\binom{x}{y}=x^2+y^2-4$

near $\bar{c}=\binom{a}{b}\neq\pm\binom{2}{0}, \pm\binom{0}{2}$

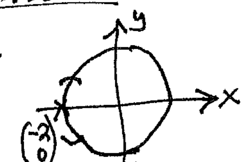


one can either express $x=+\sqrt{4-y^2}$ or $-\sqrt{4-y^2}$

and also $y=+\sqrt{4-x^2}$ or $-\sqrt{4-y^2}$

But at $x=\binom{-2}{0}$, one can only express $x=-\sqrt{4-y^2}$ in a <u>neighborhood around it</u>:

(Can't ~~write~~ decide $y=+\sqrt{4-x^2}$ vs. $-\sqrt{4-x^2}$ in any neighborhood around it)



THM (Implicit Function Thm) (More than THM 2.10.11, less than THM 2.10.14) Given $\underset{\mathbb{R}^{n+m}}{\overset{\text{open}}{\mathcal{U}}} \xrightarrow{\bar{F}} \mathbb{R}^n$ in $C^1(\mathcal{U})$

and a point $\bar{c} \in \mathcal{U}$ where $D\bar{F}(\bar{c}) : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ is <u>onto</u> (= full-rank n = surjective),

if one relabels the variables in $\mathbb{R}^{m+n}$ as $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_m \end{pmatrix}$ so that

$D\bar{F}(\bar{c}) = n\left\{ \underbrace{\left[\begin{array}{c} x_1 \cdots x_n \\ \\ \end{array}\right.}_{n} \underbrace{\left. \begin{array}{c} y_1 \cdots y_m \\ \\ \end{array} \right]}_{m} \right.$ <u>has $x_1, \dots, x_n$ as pivot columns</u>, then write $\bar{c} = \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}$

with $\bar{a} \in \mathbb{R}^n$, $\bar{b} \in \mathbb{R}^m$, there ~~are~~ open neighborhoods $\bar{a} \in \overset{\text{open}}{A} \subset \mathbb{R}^n$, $\bar{b} \in \overset{\text{open}}{B} \subset \mathbb{R}^m$

and unique $\begin{array}{ccc} B & \xrightarrow{\bar{g}} & A \\ \bar{b} & \longmapsto & \bar{g}(\bar{b}) = \bar{a} \end{array}$ with $\bar{g}$ differentiable such that $\bar{F}\begin{pmatrix} \bar{g}(\bar{y}) \\ \bar{y} \end{pmatrix} = \bar{0} \quad \forall \bar{y} \in B$,

$\boxed{\text{i.e. } \bar{x} = \bar{g}(\bar{y}) \text{ expresses } \bar{x} \text{ in terms of } \bar{y} \text{ around } \bar{c} = \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} \text{ on } F\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \bar{0}.}$

(proof in a while...)

EXAMPLES:

① $\mathbb{R}^{\overset{m}{\overset{\parallel}{1}} + \overset{n}{\overset{\parallel}{1}} = 2} \longrightarrow \mathbb{R}^{1 = n}$

$F\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 - 4$

has $JF\begin{pmatrix} x \\ y \end{pmatrix} = [2x \quad 2y]$

so $JF\begin{pmatrix} a \\ b \end{pmatrix} = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc} 2a & 2b \end{array}\right] \end{array}$ has either x or y as pivot variables if $a \neq 0$, $b \neq 0$

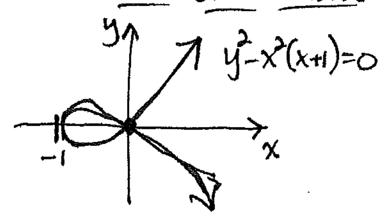
but at $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \pm 2 \\ 0 \end{pmatrix}$, $JF\begin{pmatrix} a \\ b \end{pmatrix} = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc} \pm 4 & 0 \end{array}\right] \end{array}$

only can have x as a pivot variable,
so one ~~[illegible]~~ can only deduce from Imp Fn Thm
that $\exists$ nbhds of $\begin{pmatrix} \pm 2 \\ 0 \end{pmatrix}$ where $x \left(= \pm\sqrt{4 - y^2} = g(y)\right)$ exists

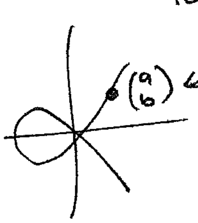
② Worse things can happen, e.g.

$\mathbb{R}^2 \longrightarrow \mathbb{R}^1$

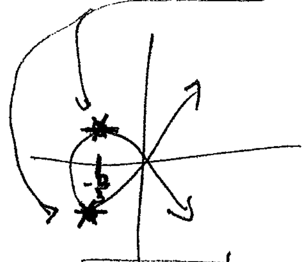
$F\begin{pmatrix} x \\ y \end{pmatrix} = y^2 - x^2(x+1) = y^2 - (x^3 + x^2)$ defines a <u>nodal</u> <u>cubic curve</u> via $F(x,y) = 0$



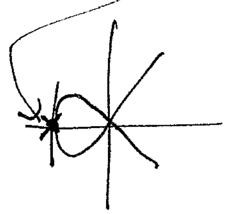
(121) Examining $JF\begin{pmatrix}x\\y\end{pmatrix} = \begin{array}{cc} x & y \\ \left[\underbrace{-(3x^2+2x)}_{=-x(3x+2)}\right. & \left. 2y \right] \end{array}$, one sees that for most $\begin{pmatrix}a\\b\end{pmatrix}$ on the curve, with $a \neq 0$, $b \neq 0$, both variables x, y are pivotal and one can write $x = g(y)$ or $y = g(x)$



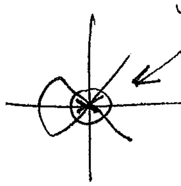
However, when $x = -\frac{2}{3}$, $y = \pm\frac{2}{3\sqrt{3}}$, $JF\begin{pmatrix}x\\y\end{pmatrix} = \pm \begin{array}{cc} x & y \\ \left[0\right. & \left.\frac{4}{3\sqrt{3}}\right] \end{array}$ and one can only write $y = g(x)$



when $y = 0$, $x = -1$, $JF\begin{pmatrix}-1\\0\end{pmatrix} = \begin{array}{cc} x & y \\ \left[1\right. & \left.0\right] \end{array}$ and one can only write $x = g(y)$



when $x = 0$, $y = 0$, $JF\begin{pmatrix}0\\0\end{pmatrix} = \underbrace{[0 \quad 0]}_{\text{not surjective!}}$, so we don't get anything from Imp. Fn. Thm!



③ (part of EXAMPLE 2.10.6) The 5-variable system $\begin{cases} x^2 - y = a \\ y^2 - z = b \\ z^2 - x = 0 \end{cases}$ has $\bar{c} = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}\begin{matrix}x\\y\\z\\a\\b\end{matrix}$ as a solution. Near $\bar{c}$, can express $\begin{pmatrix}x\\y\\z\end{pmatrix}$ in terms of $\begin{pmatrix}a\\b\end{pmatrix}$ on this sol'n set?

Here $\bar{F}: \mathbb{R}^5 \longrightarrow \mathbb{R}^3$

$$\bar{F}\begin{pmatrix}x\\y\\z\\a\\b\end{pmatrix} = \begin{pmatrix}x^2-y-a\\y^2-z-b\\z^2-x\end{pmatrix} \text{ has } JF = \begin{bmatrix} 2x & -1 & 0 & -1 & 0 \\ 0 & 2y & -1 & 0 & -1 \\ -1 & 0 & 2z & 0 & 0 \end{bmatrix}$$

$$JF(\bar{c}) = \begin{array}{c} \begin{bmatrix} 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \underbrace{x \quad y \quad z}_{\text{Yes, pivot columns!}} \quad a \quad b \end{array}$$

, so YES, by Imp. Fn. Thm.

(One could also express $\begin{pmatrix}x\\b\end{pmatrix}$ in terms of $\begin{pmatrix}y\\z\end{pmatrix}$, for example)