12/09/16

Linear Algebra and ODEs

> Reference: "Differential Equations and Dynamical Systems," by Perko. (Ch. 1)

Let's solve the very simple IVP

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = ax, & (a \in \mathbb{R}) \\ x(0) = x_0 . \end{cases}$$

By separation of variables, $x(t) = x(0)e^{at}$. Note that this is the only function that satisfies the IVP.

Now let's try

$$\begin{cases} \dot{x}_1 = a_1 x_1 \\ \dot{x}_2 = a_2 x_2 \end{cases}$$

which has the solution $x_1(t) = x_1(0)e^{a_1 t}$ and $x_2(t) = x_2(0)e^{a_2 t}$.

This can be rewritten as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and the solution can be written as $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{a_1 t} & 0 \\ 0 & e^{a_2 t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$.

In general, the system could be coupled, and we can express the (linear) system as $\dot{x} = Ax$, where $x \in \mathbb{R}^n$ and A is an $n \times n$ matrix.

Fact: Suppose that A has real eigenvalues and that they are distinct. Then the IVP $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$ has the unique solution $x(t) = e^{At}x_0$.

Example: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

After diagonalizing the matrix and making a change of coordinates, we obtain

$$x(t) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

Note: This agrees with Fact because $e^{At} = Pe^{\Lambda t}P^{-1}$ when $P^{-1}AP = \Lambda$ diagonal.