THM 2.6.22 & DEF'N 2.6.23: If $\{\bar v_1, \dots, \bar v_n\}$, $\{\bar w_1, \dots, \bar w_m\}$ are 2 bases for V, then $m = n$; $n =: \dim(V)$ dimension of V

proof: We'd like to say that the composite bijection

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\Phi_{\{\bar v\}}", "\text{a bijection}"'] \arrow[rr, bend right, "\Phi_{\{\bar w\}}^{-1} \circ \Phi_{\{\bar v\}}"'] & V \arrow[r, "\Phi_{\{\bar w\}}^{-1}", "\text{a bijection}"'] & \mathbb{R}^m \end{tikzcd}$$

Is a lin. transformation $\mathbb{R}^n \xrightarrow{\;} \mathbb{R}^m$ which is bijective, hence $m = n$.

However, 1st we should check that $\Phi_{\{\bar w\}}^{-1}$ is linear,

by noting LEMMA: If $T : V \to W$ is a bijective, linear map then $T^{-1} : W \to V$ is also linear.

proof (same as for PROP 1.3.14):

Want to check $T^{-1}(a\bar v_1 + b\bar v_2) = aT^{-1}(\bar v_1) + bT^{-1}(\bar v_2)$, but since T is bijective, this is true if and only if

$$TT^{-1}(a\bar v_1 + b\bar v_2) \overset{?}{=} T\left(aT^{-1}(\bar v_1) + bT^{-1}(\bar v_2)\right)$$

$$\begin{array}{ccc} \| & & \| \leftarrow T \text{ is linear} \\ a\bar v_1 + b\bar v_2 & & aTT^{-1}(\bar v_1) + bTT^{-1}(\bar v_2) \\ & & \| \\ & = \checkmark = & a\bar v_1 + b\bar v_2 \quad \blacksquare \end{array}$$

Thus $\Phi_{\{\bar w\}}^{-1}$ is linear,

so the composite $\Phi_{\{\bar w\}}^{-1} \circ \Phi_{\{\bar v\}} : \mathbb{R}^n \to \mathbb{R}^m$ is linear, bijective, forcing $m = n$ $\blacksquare$

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In fact, the matrix that represents

$$\begin{tikzcd} \mathbb{R}^n \arrow[rr, "\Phi_{\{\bar w\}}^{-1} \circ \Phi_{\{\bar v\}} =: \Phi_{\{\bar v \to \bar w\}} \text{ in book}"] \arrow[dr, "\Phi_{\{\bar v\}}"'] & & \mathbb{R}^n \\ & V \arrow[ur, "\Phi_{\{\bar w\}}^{-1}"'] & \end{tikzcd}$$

is the matrix that converts the coordinates w.r.t. $\{\bar v_i\}$ for a vector in V to its coordinates w.r.t. $\{\bar w_i\}$:

$$\begin{tikzcd} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \arrow[rr, "\Phi_{\{\bar w\}}^{-1} \circ \Phi_{\{\bar v\}} = \Phi_{\{\bar v \to \bar w\}}"] \arrow[dr, "\Phi_{\{\bar v\}}"'] & & \mathbb{R}^n \ni \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \arrow[dl, "\Phi_{\{\bar w\}}"] \\ & x_1\bar v_1 + \dots + x_n\bar v_n = y_1\bar w_1 + \dots + y_n\bar w_n & \end{tikzcd}$$

i.e. $\Phi_{\{\bar v \to \bar w\}}$ sends $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

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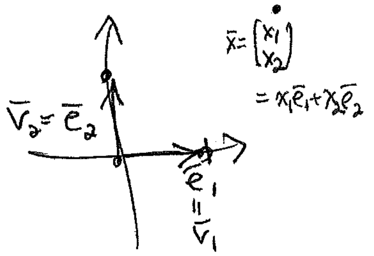
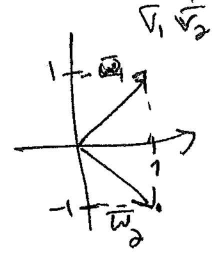
EXAMPLE: What is the matrix for $\Phi_{(\underset{\bar v_1}{\underset{\shortparallel}{\bar e_1}},\underset{\bar v_2}{\underset{\shortparallel}{\bar e_2}}) \to \left(\underset{\bar w_1}{\underset{\shortparallel}{\begin{bmatrix}1\\1\end{bmatrix}}},\underset{\bar w_2}{\underset{\shortparallel}{\begin{bmatrix}1\\-1\end{bmatrix}}}\right)} : R^2 \longrightarrow R^2$?





Uniquely write $\bar e_1 = p_{11}\bar w_1 + p_{21}\bar w_2 \;=\; \frac{1}{2}\bar w_1 + \frac{1}{2}\bar w_2$

$\bar e_2 = p_{12}\bar w_1 + p_{22}\bar w_2 \;=\; \frac{1}{2}\bar w_1 - \frac{1}{2}\bar w_2$

Then we claim $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ is the matrix for $\Phi_{(\bar e_1,\bar e_2)\to(\bar w_1,\bar w_2)}$

$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ (a somewhat bad example, due to symmetry!)

<u>PROP:</u> If $(\bar v_1, \ldots, \bar v_n)$, $(\bar w_1, \ldots, \bar w_n)$ are 2 bases for V

with $\bar v_j = \sum_{i=1}^{n} p_{ij}\,\bar w_i$, then $\Phi_{\{\bar v\}\to\{\bar w\}}$ has matrix $P = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix}$

<u>proof:</u> If $v = x_1\bar v_1 + \ldots + x_n\bar v_n$ then we want $P\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} = \begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix}$,

$= y_1\bar w_1 + \ldots + y_n\bar w_n$

so check that this works:

$$y_1\bar w_1 + \ldots + y_n\bar w_n = \sum_{j=1}^{n} x_j\bar v_j = \sum_{j=1}^{n} x_j\left(\sum_{i=1}^{n} p_{ij}\bar w_i\right)$$

$$/\!/$$

$$\sum_{i=1}^{n} y_i\bar w_i \qquad\qquad = \sum_{i=1}^{n}\underbrace{\left(\sum_{j=1}^{n} p_{ij}x_j\right)}_{\substack{\text{this must be } y_i \text{ for each } i=1,\ldots,n,\\ \text{since } \{\bar w_i\} \text{ are a basis}}}\bar w_i$$

i.e. $P\bar x = \bar y$ ∎

REMARK: There are *interesting* vector spaces V that have no finite basis, and we call them infinite-dimensional ($\dim V = \infty$)

EXAMPLES:

$$P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_d \subset \cdots \subset P := \{\text{all polynomials } p(x) = \sum_{i=0} a_i x^i\}$$

(P_d = deg $\le d$ polynomials)

$\dim =$ 1, 2, 3, …, $d+1$; $\dim P = \infty$

$\{a_0\}$ constants, $\{a_0 + a_1 x\}$ linears, $\{a_0 + a_1 x + x^2\}$ quadratics

$$P \subset \cdots \subset C^2(\mathbb{R}) \subset C^1(\mathbb{R}) \subset C^0(\mathbb{R})$$

$C^0(\mathbb{R}) = \{\text{continuous functions } f: \mathbb{R} \to \mathbb{R}\}$ (∞)

$C^1(\mathbb{R}) = \{\text{differentiable functions } f: \mathbb{R} \to \mathbb{R} \text{ with } f'(x) \text{ continuous}\}$ (∞)

$C^2(\mathbb{R}) = \{f''(x) \text{ exists, continuous}\}$ (∞)

P (∞)

§2.7 Eigenvalues & eigenvectors

– a good reason to sometimes make a change-of-basis, so that a linear map $T: V \to V$ becomes easier to understand

EXAMPLE 2.7.1 (Fibonacci numbers)

They are defined as a sequence $a_0, a_1, a_2, a_3, a_4, a_5, \ldots$ with $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8$

by $\boxed{a_0 = a_1 = 1}$ (initial conditions)

$+ \boxed{a_{n+1} = a_n + a_{n-1}}$ (recurrence)

and arise in nature in various ways (google "Fibonacci numbers & pineapples")

How fast do they grow? Roughly like a constant times φ^n where $\varphi := \frac{1+\sqrt{5}}{2}$ = golden ratio

In fact, we'll get an *exact* formula for a_n that shows this, starting by recording $\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$ as a vector, and noting that

$$\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{\text{call this } A} \begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix}$$

so

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = A \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = A^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We could understand this if we know more about the powers A^n

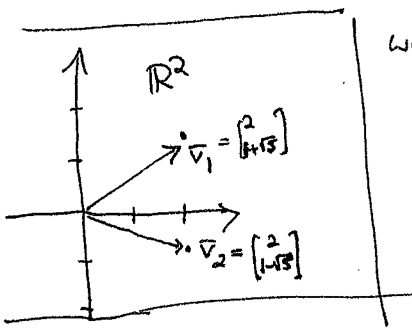
Luckily, certain vectors in $\mathbb{R}^2$ are scaled by A:

or is it luck? (Nope!)

$$A\begin{bmatrix}2\\1+\sqrt5\end{bmatrix} = \begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}2\\1+\sqrt5\end{bmatrix} = \begin{bmatrix}1+\sqrt5\\3+\sqrt5\end{bmatrix} = \frac{1+\sqrt5}{2}\begin{bmatrix}2\\1+\sqrt5\end{bmatrix}$$

$$A\begin{bmatrix}2\\1-\sqrt5\end{bmatrix} = \begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}2\\1-\sqrt5\end{bmatrix} = \begin{bmatrix}1-\sqrt5\\3-\sqrt5\end{bmatrix} = \frac{1-\sqrt5}{2}\begin{bmatrix}2\\1-\sqrt5\end{bmatrix}$$

DEF'N 2.7.2: If A is square $n\times n$ and $\bar v \in \mathbb{R}^n\setminus\{\bar 0\}$ satisfies $A\bar v = \lambda \bar v$, we say that $\bar v$ is an eigenvector for A, with eigenvalue λ.



e.g. A above has $\bar v_1 = \begin{bmatrix}2\\1+\sqrt5\end{bmatrix}$ as an eigenvector, with eigenvalue $\lambda_1 = \frac{1+\sqrt5}{2} (= \varphi)$

and $\bar v_2 = \begin{bmatrix}2\\1-\sqrt5\end{bmatrix}$ as an eigenvector, with eigenvalue $\lambda_2 = \frac{1-\sqrt5}{2}$

How did this help us with A^n?

$$A\bar v_1 = \lambda_1 \bar v_1 \Rightarrow A^n \bar v_1 = \lambda_1^n \bar v_1$$
$$A\bar v_2 = \lambda_2 \bar v_2 \Rightarrow A^n \bar v_2 = \lambda_2^n \bar v_2$$

so if we form the matrix $P = \begin{bmatrix} | & | \\ \bar v_1 & \bar v_2 \\ | & | \end{bmatrix} = \begin{bmatrix}2 & 2\\ 1+\sqrt5 & 1-\sqrt5\end{bmatrix}$, which is invertible (Why?)

then $AP = \begin{bmatrix} | & | \\ A\bar v_1 & A\bar v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \lambda_1\bar v_1 & \lambda_2\bar v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \bar v_1 & \bar v_2 \\ | & | \end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$

$$AP = P\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$$

mult. on left by P^{-1}

$$P^{-1}AP = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \quad \text{a diagonal matrix!}$$

Similarly $P^{-1}A^nP = \underbrace{(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)}_{n \text{ times}} = (P^{-1}AP)^n = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}^n = \begin{bmatrix}\lambda_1^n & 0\\ 0 & \lambda_2^n\end{bmatrix}$

(It's easy to take powers of diagonal matrices: $\begin{bmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_m\end{bmatrix}^n = \begin{bmatrix}\lambda_1^n & & 0\\ & \lambda_2^n & \\ 0 & & \ddots \ \lambda_m^n\end{bmatrix}$.)