(100) We'll take a different approach, using Leibniz's expansion

11/18/2016 > First we need sign or signature of a permutation

(DEF'N 4.8.10) $\mathrm{sgn}: \mathrm{Perm}_n \longrightarrow \{+1,-1\}$

$$\mathrm{Perm}_n = \left\{ \text{all permutations } (=\text{bijections})\ \sigma : \{1,\ldots,n\} \to \{1,\ldots,n\} \right\}, \quad \sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

$$\sigma \longmapsto \mathrm{sgn}(\sigma) := (-1)^{\#\overbrace{\{(i,j) : 1 \le i < j \le n,\ \sigma(i) > \sigma(j)\}}^{\text{inversions in } \sigma}}$$

e.g. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 6 & 4 & 2 & 3 \end{pmatrix}$ has $4+3+2 = 9$ inversions so $\mathrm{sgn}(\sigma) = (-1)^9 = -1$

PROP: For any pair $i < j$, if σ' is obtained from σ by swapping $\sigma(i), \sigma(j)$

i.e. $\sigma'(k) = \begin{cases} k & \text{if } k \ne i,j \\ \sigma(j) & \text{if } k = i \\ \sigma(i) & \text{if } k = j \end{cases}$

then $\mathrm{sgn}(\sigma') = -\mathrm{sgn}(\sigma)$

Proof:

It's very easy to check when $j = i+1$ (an adjacent swap):

e.g. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 6 & 5 & 2 \end{pmatrix}$ $\quad \sigma = \begin{pmatrix} 1 & 2 & \cdots & i & \overset{i+1}{j} & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(i) & \sigma(j) & \cdots & \sigma(n) \end{pmatrix}$ only gain exactly one inversion if $\sigma(i) < \sigma(j)$

$\sigma' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 6 & 4 & 5 & 2 \end{pmatrix}$ $\quad \sigma' = \begin{pmatrix} 1 & 2 & \cdots & i & j & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(j) & \sigma(i) & \cdots & \sigma(n) \end{pmatrix}$ or lose exactly one inversion if $\sigma(i) > \sigma(j)$

Either way $\mathrm{sgn}(\sigma') = (-1)^{\#\text{inversions in } \sigma'} = (-1)^{\#(\text{inversions in } \sigma) \pm 1} = -\mathrm{sgn}(\sigma)$.

When $j - i > 1$, mimic the swapping by $2(j-i)-1$ adjacent swaps:

e.g.

$$\left.\begin{array}{l} \left.\begin{array}{l} 1\ \overset{i}{2}\ 3\ 4\ \overset{j}{5}\ 6 \\ 1\ 2\ 3\ 5\ 4\ 6 \end{array}\right\} j-i-1 \text{ swaps} \\ \left.\begin{array}{l} 1\ 2\ 5\ 3\ 4\ 6 \\ 1\ 5\ 2\ 3\ 4\ 6 \end{array}\right\} 1 \text{ swap} \\ \left.\begin{array}{l} 1\ 5\ 3\ 2\ 4\ 6 \\ 1\ 5\ 3\ 4\ 2\ 6 \end{array}\right\} j-i-1 \text{ swaps} \end{array}\right\} 2(j-i)-1 \text{ swaps}$$

This implies $\mathrm{sgn}(\sigma') = (-1)^{2(j-i)-1}\mathrm{sgn}(\sigma) = -\mathrm{sgn}(\sigma)$ ■

DEF'N: (THM 4.8.11) For A $n\times n$ matrix, define
$$\det A := \sum_{\sigma\in \mathrm{Perm}_n} \mathrm{sgn}(\sigma)\, a_{1,\sigma(1)}\, a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

EXAMPLES: $n=1$ $\quad \det[a_{11}] = a_{11}$

$n=2$
$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} = +\, a_{11}a_{22} - a_{12}a_{21}$$
$$\underset{\sigma_1=\binom{1\,2}{1\,2}}{\begin{bmatrix} \bigcirc & \\ & \bigcirc \end{bmatrix}} \qquad \underset{\sigma_2=\binom{1\,2}{2\,1}}{\begin{bmatrix} & \bigcirc \\ \bigcirc & \end{bmatrix}}$$

$n=3$
$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix} = +a_{11}a_{22}a_{23} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$
$$\underset{\binom{123}{123}}{\begin{bmatrix} \bigcirc & \cdot & \cdot \\ \cdot & \bigcirc & \cdot \\ \cdot & \cdot & \bigcirc \end{bmatrix}} \quad \underset{\binom{123}{213}}{\begin{bmatrix} \cdot & \bigcirc & \cdot \\ \bigcirc & \cdot & \cdot \\ \cdot & \cdot & \bigcirc \end{bmatrix}} \quad \underset{\binom{123}{132}}{\begin{bmatrix} \bigcirc & \cdot & \cdot \\ \cdot & \cdot & \bigcirc \\ \cdot & \bigcirc & \cdot \end{bmatrix}}$$
$$- a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}$$
$$\underset{\binom{123}{321}}{\begin{bmatrix} \cdot & \cdot & \bigcirc \\ \cdot & \bigcirc & \cdot \\ \bigcirc & \cdot & \cdot \end{bmatrix}} \quad \underset{\binom{123}{231}}{\begin{bmatrix} \cdot & \bigcirc & \cdot \\ \cdot & \cdot & \bigcirc \\ \bigcirc & \cdot & \cdot \end{bmatrix}} \quad \underset{\binom{123}{312}}{\begin{bmatrix} \cdot & \cdot & \bigcirc \\ \bigcirc & \cdot & \cdot \\ \cdot & \bigcirc & \cdot \end{bmatrix}}$$

THM 4.8.1: $\det : \overset{\mathrm{Mat}(n,n)}{\{\text{square } n\times n \text{ matrices}\}} \longrightarrow \mathbb{R}$ has these properties (and is the unique such function, we'll see below)

$A = \begin{bmatrix} | & | & & | \\ \vec v_1 & \vec v_2 & \cdots & \vec v_n \\ | & | & & | \end{bmatrix}$

(1) det is linear in each column $\vec v_j$, i.e.
$$\det\begin{bmatrix} | & & | & & | \\ \vec v_1 & \cdots & a\vec v_j + b\vec v_j{}' & \cdots & \vec v_n \\ | & & | & & | \end{bmatrix} = a\det\begin{bmatrix} | & & | & & | \\ \vec v_1 & \cdots & \vec v_j & \cdots & \vec v_n \\ | & & | & & | \end{bmatrix} + b\det\begin{bmatrix} | & & | & & | \\ \vec v_1 & \cdots & \vec v_j{}' & \cdots & \vec v_n \\ | & & | & & | \end{bmatrix}$$
("multilinearity")

(2) swapping any 2 columns in A negates det ("alternating")

(3) $\det\begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1\end{bmatrix} = 1$ ("normalization")

proof: (3) is easy since only $\sigma = \binom{1\ 2\ \cdots\ n}{1\ 2\ \cdots\ n}$ gives a nonzero term $+1\cdot \overset{1}{\overset{\shortparallel}{a_{11}}}\,\overset{1}{\overset{\shortparallel}{a_{22}}} \cdots \overset{1}{\overset{\shortparallel}{a_{nn}}} = 1$

(2) comes from the $\mathrm{sgn}(\sigma') = -\mathrm{sgn}(\sigma)$ property when one swaps $\sigma(i), \sigma(j)$ in σ to get σ'.

(1) is easy since each term $a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}$ contains exactly one factor a_{ij} from column j (namely for $i=\sigma^{-1}(j)$), so when it is replaced by $a\cdot a_{ij} + b a_{ij}'$ the whole sum behaves the same ∎

 This gives us some easy further important properties of $\det A$

COROLLARY:

(i) If $\tilde{A}$ is obtained from A by scaling a column by c, then $\det \tilde{A} = c \det A$

(ii) If $\tilde{A}$ is obtained from A by adding c(col i) to col j, then $\det \tilde{A} = \det A$

(iii) if A has 2 equal columns, then $\det A = 0$

proof: (i) is part of multilinearity.

For (iii), 1st note that when A has 2 equal columns, $\det A = 0$ (because $\det \underset{\text{same when we swap 2 equal columns}}{A} = -\det A$)

Then for (ii),
$$\det \underbrace{\begin{bmatrix} | & & | & & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_i & \cdots & \vec{v}_j + c\vec{v}_i & \cdots & \vec{v}_n \\ | & & | & & | & & | \end{bmatrix}}_{\tilde{A}} = \det \underbrace{\begin{bmatrix} | & & | & & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_i & \cdots & \vec{v}_j & \cdots & \vec{v}_n \\ | & & | & & | & & | \end{bmatrix}}_{A} + \cancel{c \det \begin{bmatrix} | & & | & & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_i & \cdots & \vec{v}_i & \cdots & \vec{v}_n \\ | & & | & & | & & | \end{bmatrix}}^{\,0}$$

2 equal columns, so det = 0 ∎

Finally, we get...

THM 4.8.3: M $n \times n$ is invertible $\iff \det M \neq 0$

proof: Use column operations (i.e. row operations on M^T) to reduce
$$M \rightsquigarrow \tilde{M} = \begin{cases} \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} & \text{if } M \text{ is invertible} \\[2ex] \begin{bmatrix} * & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{bmatrix} & \text{if } M \text{ is not invertible} \end{cases}$$
last column zero

The above COROLLARY shows $\det M$, $\det \tilde{M}$ differ by a nonzero scalar so $\det M \neq 0 \iff \det \tilde{M} \neq 0$.

But $\det \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = 1 \neq 0$, while $\det \begin{bmatrix} * & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{bmatrix} = 0$ since each term $m_{1,\sigma(1)} \cdots m_{n,\sigma(n)}$ has a factor $m_{i,n}$ from the last column ∎

COROLLARY: The properties (1),(2),(3) of det in THM 4.8.1 characterize it uniquely as a function Mat$(n,n) \to \{\pm 1\}$

proof: They let you calculate $\det(A)$ from $\det(\tilde{A}) = 0$ or 1 if $A \overset{\text{col-reduces}}{\rightsquigarrow} \tilde{A}$ by showing how det changes with each elementary column operation ∎

RMK: This is the correct, fast ($\leq c n^3$ steps) way to compute $\det A$, versus $n!$ operations!