For (ii), if $\{\bar{v}_i\}_{i=1,\dots,k}$ are orthonormal,

given $\quad \bar{v} = c_1\bar{v}_1 + \dots + c_k\bar{v}_k$

again for $j=1,\dots,k$ take dot product with $\bar{v}_j$ to get

$$\bar{v}\bullet\bar{v}_j = c_1\underbrace{\bar{v}_1\bullet\bar{v}_j}_{0} + \dots + c_j\underbrace{\bar{v}_j\bullet\bar{v}_j}_{1} + \dots + c_k\underbrace{\bar{v}_k\bullet\bar{v}_j}_{0}$$

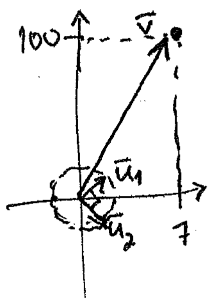
$$\Rightarrow \bar{v}\bullet\bar{v}_j = c_j \quad \blacksquare$$

11/04/2012

EXAMPLE: $\left\{ \bar{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \bar{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^2$

and hence $\bar{v} = \begin{bmatrix}7\\100\end{bmatrix} = c_1\bullet\bar{u}_1 + c_2\bullet\bar{u}_2$ where $c_1 = \bar{v}\bullet\bar{u}_1 = \begin{bmatrix}7\\100\end{bmatrix}\bullet\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$$= \frac{107}{\sqrt{2}}$$



$$c_2 = \bar{v}\bullet\bar{u}_2 = \begin{bmatrix}7\\100\end{bmatrix}\bullet\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$= \frac{-93}{\sqrt{2}}$$

NON-EXAMPLE: $E = \left\{ \text{sol'ns to } [1\ -1\ -1]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = [0] \text{ in } \mathbb{R}^3 \right\}$

had basis $\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$, but it is neither orthogonal nor orthonormal.

~~To see that $\ker(A)$ has basis $\{\bar{x}_{\ell_1},\dots,\bar{x}_{\ell_{n-p}}\}$,~~

~~note that $\ker(A) = \ker(\tilde{A}) = \{\bar{x}\in\mathbb{R}^n : \tilde{A}\bar{x} = \bar{0}\}$~~

~~which is parametrized by picking $x_{\ell_1},\dots,x_{\ell_{n-p}}$ arbitrarily~~

~~and then the values for $x_{k_1},\dots,x_{k_p}$ are forced~~

§2.5 Kernels, images, ranks

– a useful way to think about existence/uniqueness of sol'ns to linear systems

DEFIN 2.5.1:

For a linear map $T: \mathbb{R}^n \to \mathbb{R}^m$

- its <u>kernel</u> $\ker(T) := \{\bar{x} \in \mathbb{R}^n : T\bar{x} = \bar{0}\} = \{\bar{x} \in \mathbb{R}^n :$ any sol'n to $T\bar{y}=b$ gives rise to new sol'n $T(\bar{x}+\bar{y})=b\}$
- its <u>image</u> $\operatorname{img}(T) := \{\bar{b} \in \mathbb{R}^m : \exists \bar{x} \in \mathbb{R}^n$ with $T\bar{x} = \bar{b}\}$
$= \{\bar{b} \in \mathbb{R}^m : T\bar{x} = \bar{b}$ has <u>any</u> sol'ns$\}$

If T is represented by the $m \times n$ matrix $[T] = \begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{bmatrix}$

then $\ker(T) := \left\{ \bar{x} \in \mathbb{R}^n : \bar{0} = T\bar{x} = \begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\bar{v}_1 + \dots + x_n\bar{v}_n \right\}$

$= \{\bar{x} \in \mathbb{R}^n : x_1, \dots, x_n$ give a <u>lin. dependence on the columns of $[T]$</u>$\}$

and $\operatorname{img}(T) := \left\{ T\bar{x} = \begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\bar{v} + \dots + x_n\bar{v}_n \text{ for } \bar{x} \in \mathbb{R}^n \right\}$

$= \operatorname{span}(\bar{v}_1, \dots, \bar{v}_n) =$ the <u>span of the columns</u> of $[T]$
$=:$ the column space of $[T]$

It's easy to see both $\underset{\mathbb{R}^n}{\ker(T)}$, $\underset{\mathbb{R}^m}{\operatorname{im}(T)}$ are <u>subspaces</u>. How to find <u>bases</u> for them?
($\ker(T) \subset \mathbb{R}^n$, $\operatorname{im}(T) \subset \mathbb{R}^m$)

<u>THM 2.5.4+2.5.6</u>: If the linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ has matrix $A = \begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{bmatrix}$ that row-reduces with pivotal columns $k_1, \dots, k_p$, then non-pivotal columns $l_1, \dots, l_{n-p}$

(a) $\operatorname{img}(T)$ has basis $\{\bar{v}_{k_1}, \dots, \bar{v}_{k_p}\}$ = the pivotal columns of A

(b) $\ker(T)$ has basis $\{\bar{x}_{l_1}, \dots, \bar{x}_{l_{n-p}}\}$ where $\bar{x}_l \in \mathbb{R}^n$ is the solution

to $A\bar{x}_{l_j} = \bar{0}$ having $x_{l_j} = 1$
$x_{l_m} = 0$ if $l_m \neq l_j$
(and entries x_{k_r} determined uniquely by this)

EXAMPLE: $T: \mathbb{R}^5 \to \mathbb{R}^2$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \longmapsto A\bar{x} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

has $A \overset{\text{row-reduce}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \rightsquigarrow \begin{array}{c} \begin{matrix} k_1 & l_1 & k_2 & l_2 & l_3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \end{array}$

pivotal columns 1, 3
$ \overset{\shortparallel}{k_1} \; \overset{\shortparallel}{k_2}$

Hence $\operatorname{img}(T)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

col 1 of A $= k_1$; col 3 of A, $3 = k_2$

and $\ker(T)$ has basis given by the solutions $\bar{x}_2, \bar{x}_4, \bar{x}_5$
to $A\bar{x} = \bar{0}$ with $\bar{x}_2 = \begin{bmatrix} ? \\ 1 \\ ? \\ 0 \\ 0 \end{bmatrix}\begin{matrix} \\ l_1 \\ \\ l_2 \\ l_3 \end{matrix}$, $\bar{x}_4 = \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \\ 0 \end{bmatrix}\begin{matrix} \\ l_1 \\ \\ l_2 \\ l_3 \end{matrix}$, $\bar{x}_5 = \begin{bmatrix} ? \\ 0 \\ ? \\ 0 \\ 1 \end{bmatrix}\begin{matrix} \\ l_1 \\ \\ l_2 \\ l_3 \end{matrix}$

$$\Rightarrow \bar{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{x}_4 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{x}_5 = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

---

proof of THM 2.5.4, 2.5.6:

To see $\operatorname{img}(A)$ has ~~span of~~ basis the pivotal columns of A

e.g.
$$\tilde{A} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & a & 0 & b & c & 0 & d \\ 0 & 0 & 1 & e & f & 0 & g \\ 0 & 0 & 0 & 0 & 0 & 1 & h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\Big\Uparrow$ row-reduce

$$A = \begin{bmatrix} | & | & | & | & | & | & | \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \bar{a}_4 & \bar{a}_5 & \bar{a}_6 & \bar{a}_7 \\ | & | & | & | & | & | & | \end{bmatrix}$$

note first that $\operatorname{img}(A) =$ span of <u>all</u> columns of $A = \{A\bar{x} = x_1\bar{a}_1 + \dots + x_n\bar{a}_n\}$

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However in $\tilde{A}$ there are dependencies among the columns that let one express each non-pivotal column in terms of the preceding pivotal ones,

e.g.
$$\overset{\tilde{a}_5}{\begin{bmatrix} c \\ f \\ i \\ 0 \end{bmatrix}} = c \overset{\tilde{a}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + f \overset{\tilde{a}_3}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$$

i.e. $\tilde{A}\bar{x} = \bar{0}$ where $\bar{x} = \begin{bmatrix} -c \\ 0 \\ -f \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

hence the same $\bar{x}$ satisfies $A\bar{x} = \bar{0}$, i.e. $\bar{a}_5 = c\bar{a}_1 + f\bar{a}_3$.

Also in $\tilde{A}$ the pivotal columns look like $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_p$ and hence have no nontrivial lin. dependence, so the same holds for the corresponding columns in A.

To see what a basis for $\ker(A)$ is,

note $\ker(A) = \ker(\tilde{A}) = \{\bar{x} \in \mathbb{R}^n : \tilde{A}\bar{x} = \bar{0}\}$

$\ker(A) = \{\bar{x} \in \mathbb{R}^n : A\bar{x} = \bar{0}\}$

parametrize all such $\bar{x}$ by picking $x_{l_1}, \dots, x_{l_{n-p}}$ arbitrarily and then $x_{k_1}, \dots, x_{k_p}$ are determined

e.g.
$$\tilde{A} = \begin{array}{c} \begin{array}{ccccccc} & l_1 & & l_2 & l_3 & & l_4 \\ & 2 & & 4 & 5 & & 7 \end{array} \\ \begin{bmatrix} 1 & a & 0 & b & c & 0 & d \\ 0 & 0 & 1 & e & f & 0 & g \\ 0 & 0 & 0 & 0 & 0 & 1 & h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ \vdots \\ x_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \bar{x} = \begin{bmatrix} -ax_2 - bx_4 - cx_5 - dx_7 \\ x_2 \\ -ex_4 - fx_5 - gx_7 \\ x_4 \\ x_5 \\ -hx_7 \\ x_7 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} ? \\ 1 \\ ? \\ 0 \\ 0 \\ ? \\ 0 \end{bmatrix}}_{\bar{x}_2, \text{ by defn}} + x_4 \underbrace{\begin{bmatrix} ? \\ 0 \\ ? \\ 1 \\ 0 \\ ? \\ 0 \end{bmatrix}}_{\bar{x}_4} + x_5 \underbrace{\begin{bmatrix} ? \\ 0 \\ ? \\ 0 \\ 1 \\ ? \\ 0 \end{bmatrix}}_{\bar{x}_5} + x_7 \underbrace{\begin{bmatrix} ? \\ 0 \\ ? \\ 0 \\ 0 \\ ? \\ 1 \end{bmatrix}}_{\bar{x}_7}$$

11/07/2016 > i.e. $\{\bar{x}_2, \bar{x}_4, \bar{x}_5, \bar{x}_7\}$ give a basis for $\ker(A)$.