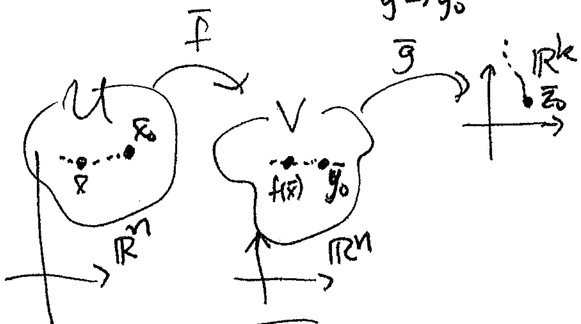
THM 1.5.24 (limit of composition) If we have $\underset{\mathbb{R}^n}{\overset{\cap}{U}} \xrightarrow{\bar{f}} \underset{\mathbb{R}^m}{\overset{\cap}{V}} \xrightarrow{\bar{g}} \mathbb{R}^k$

and both $\lim_{\bar{x}\to\bar{x}_0} f(\bar{x}) = \bar{y}_0$, $\lim_{\bar{y}\to\bar{y}_0} g(\bar{y}) = \bar{z}_0$ exist, then $\lim_{\bar{x}\to\bar{x}_0} (\bar{g}\circ\bar{f})(\bar{x}) = \bar{z}_0$ exists too.



Proof: not hard; read it in book ∎

Continuity also proceeds as one might expect..

DEF'N: $\underset{\mathbb{R}^n}{\overset{\cap}{X}} \xrightarrow{\bar{f}} \mathbb{R}^m$ is continuous at $\bar{x}_0 \in X$ if $\lim_{\bar{x}\to\bar{x}_0} f(\bar{x}) = f(\bar{x}_0)$

i.e. $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall \bar{x} \in X$ with $|\bar{x} - \bar{x}_0| < \delta$ one has $|f(\bar{x}) - f(\bar{x}_0)| < \epsilon$.

$\bar{f}$ is continuous on X if it is continuous at every $\bar{x}_0 \in X$.

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THM 1.5.28 $\bar{f}, \bar{g} : \underset{\mathbb{R}^n}{\overset{\cap}{U}} \to \mathbb{R}^m$, $h : U \to \mathbb{R}$ all continuous at $\bar{x}_0$

$\Rightarrow$
1. $\bar{f} + \bar{g}$ cont. at $\bar{x}_0$
2. $h\bar{f}$ cont. at $\bar{x}_0$
3. $\frac{\bar{f}}{h}$ cont. at $\bar{x}_0$ if $h(\bar{x}_0) \neq 0$
4. $\bar{f} \cdot \bar{g}$ cont. at $\bar{x}_0$
5. (..some bounded statement...)

easily follow from the limit laws

THM 1.5.29: $\underset{\mathbb{R}^n}{\overset{\cap}{U}} \xrightarrow{f} \underset{\mathbb{R}^m}{\overset{\cap}{V}} \xrightarrow{g} \mathbb{R}^k$ with $\bar{f}$ cont. at $\bar{x}_0$, $\bar{g}$ cont. at $f(\bar{x}_0)$

then $\bar{g}\circ\bar{f}$ cont. at $\bar{x}_0$

COROLLARY 1.5.30: Polynomial functions $f : \mathbb{R}^n \to \mathbb{R}$ are continuous on $\mathbb{R}^n$, and rational functions $f(\bar{x}) = \frac{g(\bar{x})}{h(\bar{x})}$ (so g, h polynomial) are continuous at $\bar{x}_0 \in \mathbb{R}^n$ with $h(\bar{x}_0) \neq 0$.

Infinite sums – work similarly ~~[illegible]~~ in $\mathbb{R}^n$ as in $\mathbb{R}$

DEF'N: Say $\sum_{i=1}^{\infty} \bar{a}_i$ converges if the sequence $(\bar{s}_n)_{n=1}^{\infty}$

where $\bar{s}_n := \sum_{i=1}^{n} \bar{a}_i = \bar{a}_1 + \bar{a}_2 + \dots + \bar{a}_n$

converges in $\mathbb{R}^n$

PROP 1.5.34 In $\mathbb{R}^n$,

$$\sum_{i=1}^{\infty} |\bar{a}_i| \text{ convergent} \Rightarrow \sum_{i=1}^{\infty} \bar{a}_i \text{ converges}$$

(in which case you say $\overset{\text{DEF'N:}}{\ } \sum_{i=1}^{\infty} \bar{a}_i$ converges absolutely)

Proof: Assume $\sum_{i=1}^{\infty} |\bar{a}_i|$ converges.

To show $\sum_{i=1}^{\infty} \bar{a}_i$ converges, where $\bar{a}_i = \begin{bmatrix} a_{1,i} \\ a_{2,i} \\ \vdots \\ a_{n,i} \end{bmatrix}$

we can show $(\bar{s}_n)$ converges ~~pointwise~~ coordinatewise,

so it's enough to show $\sum_{i=1}^{\infty} a_{1,i}$ converges.

But $\sum_{i=1}^{\infty} |a_{1,i}|$ converges by comparison to $\sum_{i=1}^{\infty} |\bar{a}_i|$ since $|a_{1,i}| \le |\bar{a}_i| = \sqrt{a_{1,i}^2 + \dots + a_{n,i}^2}$

Provable via ϵ-δ's: If $0 \le a_n \le b_n$ and $\sum_n b_n$ converges the $\sum_n a_n$ converges

so it's enough to show (leftover from Chapter 0!)

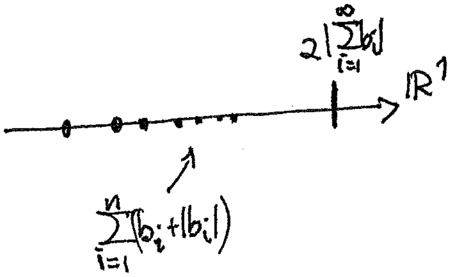
THM 0.5.8 (The $n=1$ case)

$$\sum_{i=1}^{\infty} |b_i| \text{ converges} \Rightarrow \sum_{i=1}^{\infty} b_i \text{ converges}$$

This is slightly tricky: Write $\sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} \underbrace{(b_i + |b_i|)}_{\text{nonnegative}} + \underbrace{\left(-\sum_{i=1}^{\infty} |b_i|\right)}_{\text{converges by hypothesis}}$

But also $\sum_{i=1}^{n} \overbrace{b_i + |b_i|}^{\text{nonnegative}}$ are bounded in $\mathbb{R}^1$,

since $\left|\sum_{i=1}^{n} (b_i + |b_i|)\right| \le \sum_{i=1}^{n} |b_i + |b_i|| \le \sum_{i=1}^{n} |b_i| + |b_i| = 2\sum_{i=1}^{n} |b_i| \le 2\sum_{i=1}^{\infty} |b_i|$



So $\sum_{i=1}^{\infty} (b_i + |b_i|)$ converges in $\mathbb{R}^1$, hence $\sum_{i=1}^{\infty} b_i$ does also ∎

An interesting family of examples...

PROP 1.5.37: For $n\times n$ A with $|A|<1$,

the series $I+A+A^2+\dots$ converges to a matrix $S=(I-A)^{-1}$.
i.e. S is a 2-sided inverse for $I-A$

(like $r\in\mathbb{R}$ with $|r|<1$ has $1+r+r^2+r^3+\dots$ converging to $\frac{1}{1-r}$)
geometric series

proof: (View $n\times n$ matrices like vectors; previous convergence results apply!)

Note the partial sum $S_k := I+A+A^2+\dots+A^k$

$$\text{has } S_k(I-A) = (I+A+A^2+\dots+A^k)(I-A)$$
$$= I+A+A^2+\dots+A^k - A - A^2 - \dots - A^k - A^{k+1} = I-A^{k+1}$$

1st note that on HW you show $|A^k| \le |A|^k$, so $|A|<1$

$\Rightarrow$ the series does converge absolutely to a matrix S

Now $S(I-A) = \left(\lim_{k\to\infty} S_k\right)\cdot(I-A) = \lim_{k\to\infty}\left(S_k(I-A)\right) = \lim_{k\to\infty}(I-A^{k+1}) = I - \lim_{k\to\infty} A^{k+1} = I - 0 = I.$

Think about this: Why does this follow from $\lim_{k\to\infty} \vec{a}_k \bullet \vec{b}_k = \left(\lim_{k\to\infty}\vec{a}_k\right)\bullet\left(\lim_{k\to\infty}\vec{b}_k\right)$?

Similar argument shows $(I-A)S=I$, so $S=(I-A)^{-1}$ ∎

COR 1.5.39: Within $n\times n$ matrices, thought of as $\mathbb{R}^{n^2}$, the set $\{\text{invertible matrices}\}=:\mathcal{U}$ is open.

proof: Given B invertible, so $B\in\mathcal{U}$,
we'll show that the ball of radius $\epsilon=\frac{1}{|B^{-1}|}$ around B all lies in $\mathcal{U}$.

Given H in this ball, i.e. $|H|<\frac{1}{|B^{-1}|}$, then $|\underbrace{B^{-1}H}_{A:=}| \le |B^{-1}|\cdot|H|<1$

and hence $I+\underbrace{B^{-1}H}_{A}$ is invertible, and we claim $(I+B^{-1}H)^{-1}B^{-1} = (B+H)^{-1}$:

clear already since $B+H = B(I+B^{-1}H)$

$(I+B^{-1}H)^{-1}B^{-1}\cdot(B+H) = (I+B^{-1}H)^{-1}(I+B^{-1}H) = I$ ✓

$(B+H)(I+B^{-1}H)^{-1}B^{-1} = B(I+B^{-1}H)(I+B^{-1}H)^{-1}B^{-1} = B\cdot B^{-1} = I$ ✓

Thus $B+H$ lies in $\mathcal{U}$ for all such H ∎