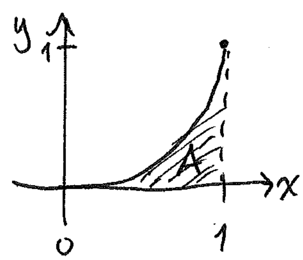
Math 3593H Honors Math II
Midterm exam 2, Thursday April 6, 2017

Instructions:
50 minutes, closed book, no electronic devices, but an 8.5 × 11 page of notes is fine. There are four problems, worth 25 points each.

1. (25 points) Find the coordinates $(\bar{x}, \bar{y})$ for the centroid (=center of gravity) of the subset $A \subset \mathbb{R}^2$ bounded above by the curve $y = x^3$, bounded below by the x-axis, bounded on the right by the line $x = 1$.

Half credit for setting up the two integrals, half for evaluating them. (Hint: sketch A first!)



$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{\text{area of } A}\begin{pmatrix}\int_A x\,|dx\,dy|\\ \int_A y\,|dx\,dy|\end{pmatrix}$$

$$= \frac{1}{\int_0^1 x^3\,dx}\begin{pmatrix}\int_{x=0}^{x=1}\int_{y=0}^{y=x^3} x\,dy\,dx\\ \int_{x=0}^{x=1}\int_{y=0}^{y=x^3} y\,dy\,dx\end{pmatrix}$$

$$= \frac{1}{\left[\frac{x^4}{4}\right]_0^1}\begin{pmatrix}\int_0^1 x\cdot\left[y\right]_0^{x^3} dx\\ \int_0^1 \left[\frac{y^2}{2}\right]_0^{x^3} dx\end{pmatrix}$$

$$= \frac{1}{1/4}\begin{pmatrix}\int_0^1 x^4\,dx\\ \int_0^1 \frac{x^6}{2}\,dx\end{pmatrix} = 4\begin{pmatrix}\left[\frac{x^5}{5}\right]_0^1\\ \left[\frac{x^7}{14}\right]_0^1\end{pmatrix} = \begin{pmatrix}4/5\\ 2/7\end{pmatrix}$$

2. For these two problems, set up an integral which would correctly calculate the desired quantity, but **DO NOT** evaluate it.

(i) (12 points) Arc length of the curve $C = \left\{ \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix} : 0 \le t \le 1 \right\}$

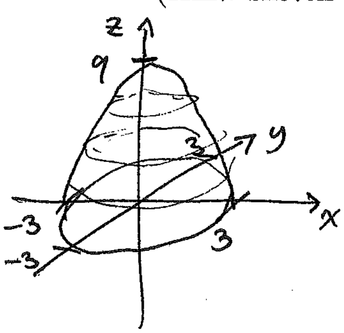
$$\bar{\gamma}(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}, \quad D\bar{\gamma}(t) = \bar{\gamma}'(t) = \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix} \qquad D\bar{\gamma}(t)^T D\bar{\gamma}(t) = [1 \ 2t \ 3t^2] \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix} = 1 + 4t^2 + 9t^4$$

$$\text{arc length}(C) = \int_0^1 \sqrt{1+4t^2+9t^4}\, dt$$

(ii) (13 points) Surface area for the part of the paraboloid

$$z = 9 - (x^2 + y^2)$$

lying above the xy-plane, that is, where $z \ge 0$.
(Hint: sketch that part of the paraboloid first!)



$$\bar{\gamma}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 9-(x^2+y^2) \end{pmatrix}, \quad D\bar{\gamma}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2x & -2y \end{bmatrix}$$

$$D\bar{\gamma}\begin{pmatrix} x \\ y \end{pmatrix}^T D\bar{\gamma}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 0 & -2x \\ 0 & 1 & -2y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2x & -2y \end{bmatrix}$$

$$= \begin{bmatrix} 1+4x^2 & 4x^2 \\ 4y^2 & 1+4y^2 \end{bmatrix}, \quad \det D\bar{\gamma}\begin{pmatrix} x \\ y \end{pmatrix}^T D\bar{\gamma}\begin{pmatrix} x \\ y \end{pmatrix} = (1+4x^2)(1+4y^2) - (4x^2)(4y^2)$$
$$= 1 + 4x^2 + 4y^2$$

$$\text{surface area} = \int_{x=-3}^{x=+3} \int_{y=-\sqrt{9-x^2}}^{y=+\sqrt{9-x^2}} \sqrt{1+4x^2+4y^2}\, dy\, dx$$

3. Prove or disprove in each case.

(i) (6 points) Simpson's numerical approximation using 100 subintervals for the integral $\int_0^1 (x^3+2)dx$ will have value $\frac{9}{4}$.

Yes, since Simpson is exact for cubic polynomials, so it should give $\int_0^1 (x^3+2)dx = \left[\frac{x^4}{4}+2x\right]_0^1 = \frac{9}{4}$ on the nose.

(ii) (6 points) The indicator function $f(x) = 1_{\mathbb{Q}}(x)$ for the rational numbers inside $\mathbb{R}^1$ is Lebesgue-integrable, with Lebesgue integral $\int_{\mathbb{R}} f(x)|d^1x| = 0$.

Yes, since $\mathbb{Q}$ has measure zero, so $f(x) \underset{a.e.}{=} g(x)$ where $g(x)=0 \ \forall x \in \mathbb{R}$, and $g(x)$ is obviously Riemann-integrable with $\int_{\mathbb{R}} g(x)|dx| = 0$.

(iii) (6 points) The subset $A := [0,1] - \mathbb{Q}$, that is, the *irrational* numbers in the interval $[0,1]$, has measure zero.

No, since (for example), if A had measure zero, then since $\mathbb{Q}\cap[0,1]$ has measure zero, we'd conclude $[0,1] = A \cup (\mathbb{Q}\cap[0,1])$ has measure zero. But we know this is false: $[0,1]$ is pavable, with $\int_{\mathbb{R}} 1_{[0,1]}(x)|d^1x| = 1 \neq 0$

(iv) (7 points) This function $\mathbb{R}^1 \xrightarrow{f} \mathbb{R}$ is Riemann-integrable:

$$f(x) = \begin{cases} x^2 & \text{for } x \in \mathbb{Q}\cap[0,1], \\ 0 & \text{otherwise.} \end{cases}$$

No, since it is discontinuous at every $x \in (0,1]$:

Having fixed x, if $x \in \mathbb{Q}$ $\exists\, y \in [0,1]\setminus\mathbb{Q}$ arbitrarily near x having $|f(x)-f(y)| > \frac{x^2}{2} =: \epsilon$

and if $x \notin \mathbb{Q}$ $\exists\, y \in \mathbb{Q}\cap[0,1]$ arbitrarily near x having $|f(x)-f(y)| > \frac{x^2}{2} =: \epsilon$

Thus f is discontinuous on a set $(0,1]$ which does not have measure zero, so it cannot be Riemann-integrable.

4. (25 points) Prove that when n is *odd*, then every $n \times n$ matrix A which is *antisymmetric*, meaning $A^\top = -A$, will have $\det(A) = 0$.

Partial credit given for only verifying the special cases $n = 1$ and $n = 3$.

$$A^T = -A$$

$$\Rightarrow \det(A^T) = \det(-A)$$

$$\det(A^T) = \det(A)$$

$$\det(-A) = (-1)^n \det(A)$$ since $-A$ is obtained from A by negating all n of its collumns

$$(-1)^n \det(A) = -\det(A)$$ since n is <u>odd</u>

$$\Rightarrow \det(A) = -\det(A)$$

$$\Rightarrow 2\det(A) = 0$$

i.e. $\det(A) = 0$