Math 3593H Honors Math II
Quiz 3, Thursday March 23, 2017

Instructions:
20 minutes, closed book, no electronic devices,
but an 8.5×11 page of notes is OK.
There are three problems, worth a total of 20 points.

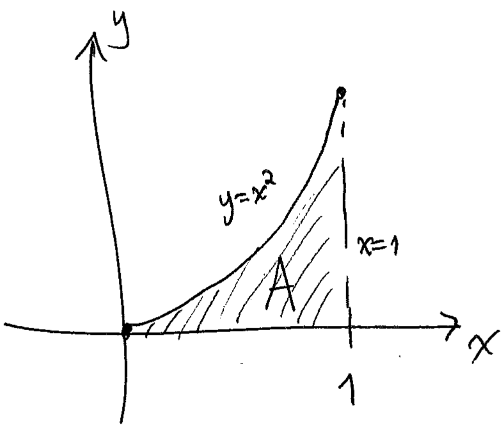
1. (9 points)
Let $A \subset \mathbb{R}^2$ be the region bounded

- above by the parabola $y = x^2$,
- below by the x-axis,
- on the right by the vertical line $x = 1$.

Compute

$$\int_A xy\,|dxdy| \left(= \int_{\mathbb{R}^2} xy \cdot 1_A(x,y)\,|dxdy| \right).$$

(Hint: it's always a good idea to sketch A first.)



$$\int_A xy\,|dx\,dy| = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=x^2} xy\,dy \right) dx$$

Fubini's Thm.

$$= \int_{x=0}^{x=1} x \left[\frac{y^2}{2} \right]_{y=0}^{y=x^2} dx$$

$$= \int_{x=0}^{x=1} \frac{x^5}{2}\,dx$$

$$= \frac{1}{2} \left[\frac{x^6}{6} \right]_{x=0}^{x=1} = \frac{1}{12}$$

2. (6 points)
What is the volume of the image of the unit cube $Q = [0,1]^3 \subset \mathbb{R}^3$ under the linear transformation $\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^3$ defined by

$$T(\mathrm{e}_1) = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}, \quad T(\mathrm{e}_2) = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}, \quad T(\mathrm{e}_3) = \begin{bmatrix} 16 \\ -73 \\ 3 \end{bmatrix} ?$$

$T(\bar{x}) = A\bar{x}$ where $A = \begin{bmatrix} 2 & 5 & 16 \\ 5 & 2 & -73 \\ 0 & 0 & 3 \end{bmatrix}$

so $\mathrm{vol}_3\, T(Q) = \det A = \det \begin{bmatrix} 2 & 5 & 16 \\ 5 & 2 & -73 \\ 0 & 0 & 3 \end{bmatrix} = (2\cdot 2 - 5\cdot 5)\cdot 3 = -21\cdot 3 = -63$

3. (5 points)
Prove or disprove: the subset $\mathbb{Q}^2 \subset \mathbb{R}^2$ consisting of all points with rational coordinates has measure zero.

proof: $\mathbb{Q}$ is countable, as discussed in book and lecture,

hence $\mathbb{Q}^2 = \bigcup_{q \in \mathbb{Q}} \underbrace{\left\{ \binom{q}{y} \in \mathbb{Q}^2 : y \in \mathbb{Q} \right\}}_{\text{this is in bijection with } \mathbb{Q} \text{ itself via } \binom{q}{y} \mapsto y, \text{ so countable}}$

Thus $\mathbb{Q}^2$ is a countable union of countable sets, so also countable (as discussed in lecture).

Hence $\mathbb{Q}^2$ is a countable union of points, which each have measure zero, and hence a countable union of sets of measure zero.

Therefore $\mathbb{Q}^2$ itself has measure zero (as discussed in lecture).