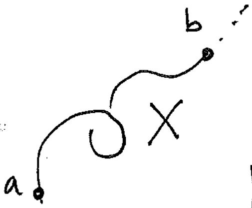
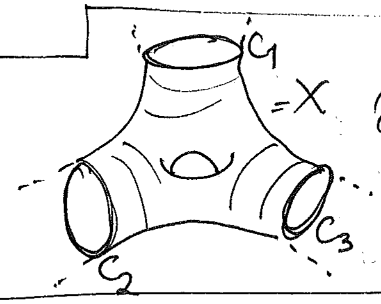
EXAMPLES:

① $k=1$ curves $X \subset C$ a smooth curve with endpoints



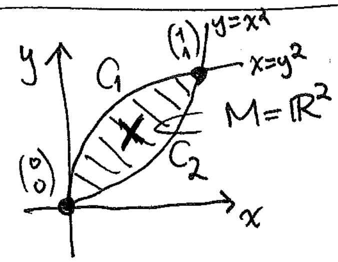
$\partial_C X = \{a,b\} = \partial^s_C X$

② $k=2$ surfaces $X \subset S$ a smooth surface with boundary curves $C_1, C_2, \ldots, C_r$



$\partial_S X = C_1 \cup C_2 \cup \cdots \cup C_r = \partial^s_S X$

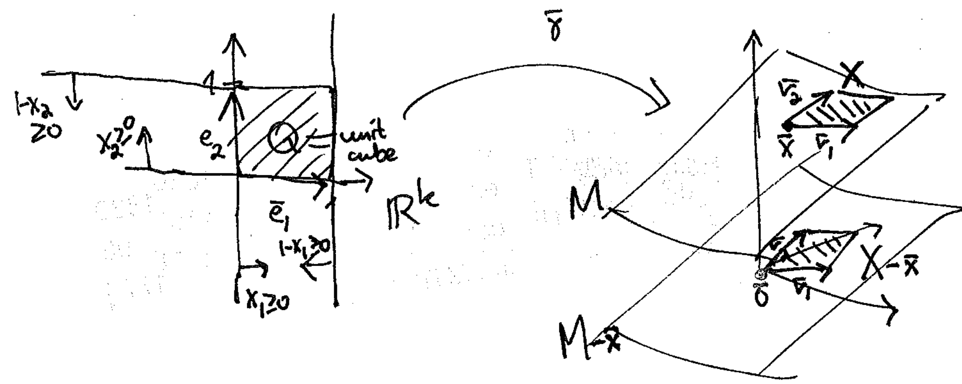
③



$\partial_M X = C_1 \cup C_2$

$\partial^s_M X = C_1 \cup C_2 - \{\binom{0}{0}, \binom{1}{1}\}$ (Why?)

④ Parallelepipeds (EXAMPLE 6.6.10) $X = P_{\bar{x}}(\bar{v}_1, \ldots, \bar{v}_k) \subset \mathbb{R}^n$ are always pieces-with-boundary inside the k-dim'l manifold $M = \underbrace{\{\bar{x} + t_1\bar{v}_1 + \cdots + t_k\bar{v}_k : t_i \in \mathbb{R}\}}_{\text{the (affine) k-dim'l subspace containing } X}$



Not hard to check X is compact (=closed, bounded).

For each $(k-1)$-dim'l "face" of X, to get the appropriate functions

$$\mathbb{R}^n \xrightarrow{\binom{\bar{f}}{g}} \mathbb{R}^{n-(k-1)}$$
$$\bar{y} \longmapsto \begin{pmatrix} \bar{f}(\bar{y}) \\ g(\bar{y}) \end{pmatrix}$$

it's probably easier to work with $X - \bar{x}$ $= P_{\bar{0}}(\bar{v}_1, \ldots, \bar{v}_k)$,

since then if one extends $\bar{v}_1, \ldots, \bar{v}_k$ to a basis $\bar{v}_1, \ldots, \bar{v}_k, \bar{v}_{k+1}, \ldots, \bar{v}_n$ for $\mathbb{R}^n$, then the linear isomorphism

$$\begin{aligned} \mathbb{R}^n &\xrightarrow{T} \mathbb{R}^n \\ \bar{v}_1 &\longmapsto \bar{e}_1 \\ &\vdots \\ \bar{v}_k &\longmapsto \bar{e}_k \\ \bar{v}_{k+1} &\longmapsto \bar{e}_{k+1} \\ &\vdots \\ \bar{v}_n &\longmapsto \bar{e}_n \end{aligned}$$

lets one define $\bar{f}(\bar{y}) = \begin{pmatrix} T(\bar{y})_{k+1} \\ \vdots \\ T(\bar{y})_n \end{pmatrix}$

to cut out $M - \bar{x}$ as $\bar{f}^{-1}(\bar{0})$,

and cut out various faces/half spaces via $g(\bar{y}) = T(\bar{y})_i \geq 0$ for $i = 1, 2, \ldots, k$ or $1 - T(\bar{y}_i) \geq 0$

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⑤ See NON-EXAMPLES 6.6.8, 6.6.9 in book!

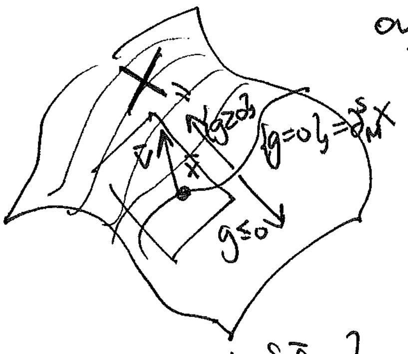
(111) Given a piece-with-boundary $X \subset M$ a manifold, here's how to orient the $(k-1)$-dim'l manifold $\partial_M^s X$ given an orientation Ω on X.

DEF'N 6.6.14 / 6.6.15: For $\bar{x} \in \partial_M^s X$, say a tangent vector $\bar{v} \in T_{\bar{x}} M$ is

outward-pointing if $\underbrace{[Dg(\bar{x})]}_{1\times k}\underbrace{\bar{v}}_{k\times 1} < 0$

inward-pointing if $[Dg(\bar{x})]\bar{v} > 0$

(neither if $[Dg(\bar{x})]\bar{v} = 0$).

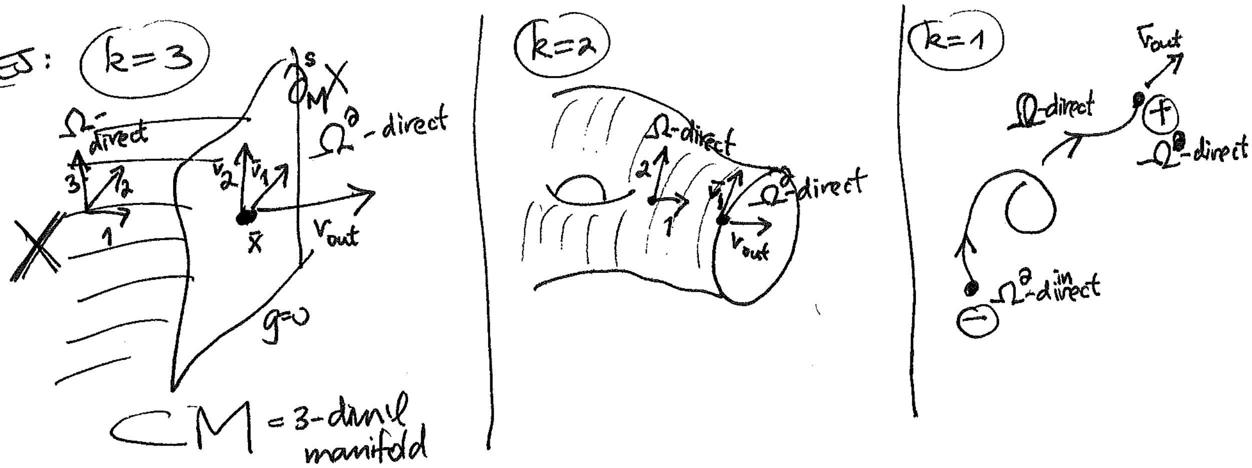


$M = \{\bar{f} = \bar{0}\}$

Then define the induced orientation for $\partial_M^s X$ by

$$\Omega^{\partial}_{\bar{x}}(\bar{v}_1, \ldots, \bar{v}_{k-1}) := \Omega_{\bar{x}}(\bar{v}_{out}, \bar{v}_1, \ldots, \bar{v}_k)$$ where $\bar{v}_{out}$ is any outward-pointing vector in $T_{\bar{x}} M$

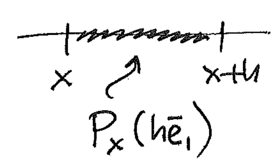
EXAMPLES:



§6.7 Exterior derivative

Stokes will say $\int_X d\varphi = \int_{\partial X} \varphi$ for some operation $A^k(\mathbb{R}^n) \xrightarrow{d} A^{k+1}(\mathbb{R}^n)$, $\omega \longmapsto d\omega$,

capturing $\int_a^b \underbrace{df}_{=f'(x)dx} = +f(b) - f(a)$ where $f'(x) := \lim_{h\to 0} \frac{1}{h^1}\big(+f(x+h) - f(x)\big)$



$d\varphi$ will be defined by a similar limit.

Given a parallelepiped $P_{\bar{x}}(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k) = P$

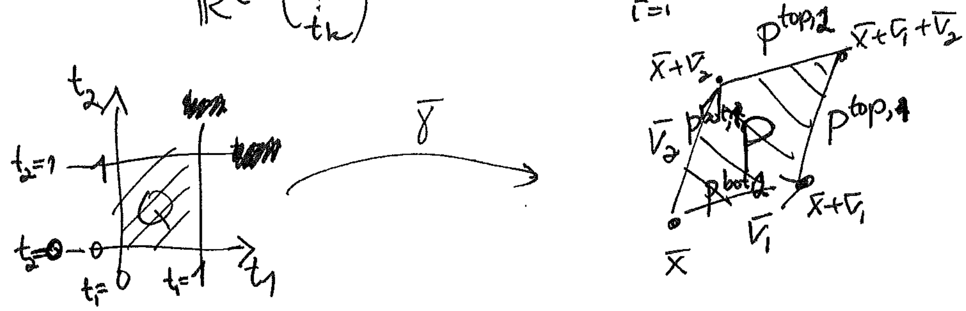
omit $\bar{v}_i$

define its i^{th} top face $\quad P^{top,i}_{\bar{x}}(\bar{v}_1, \dots, \bar{v}_k) := P_{\bar{x}+\bar{v}_i}(\bar{v}_1, \dots, \hat{\bar{v}}_i, \dots, \bar{v}_k) = P \cap \bar{\gamma}(t_i = 1)$

i^{th} bottom face $\quad P^{bot,i}_{\bar{x}}(\bar{v}_1, \dots, \bar{v}_k) := P_{\bar{x}}(\bar{v}_1, \dots, \hat{\bar{v}}_i, \dots, \bar{v}_k) = P \cap \bar{\gamma}(t_i = 0)$

where $\bar{\gamma}$ is the parametrization

$$
\begin{array}{ccc}
Q = [0,1]^k & \xrightarrow{\bar{\gamma}} & P_{\bar{x}}(\bar{v}_1, \dots, \bar{v}_k) \\
\cap & & \\
\mathbb{R}^k \ \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} & \longmapsto & \bar{x} + \sum_{i=1}^{k} t_i \bar{v}_i
\end{array}
$$



<u>DEF'N 6.7.1</u>: For $U \overset{open}{\subset} \mathbb{R}^n$, define $A^k(U) \xrightarrow{d} A^{k+1}(U)$

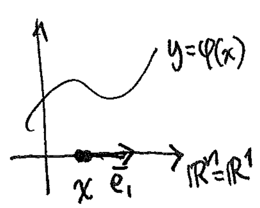
by $d\varphi(P_{\bar{x}}(\bar{v}_1, \dots, \bar{v}_{k+1}))$

$$
:= \lim_{h \to 0} \frac{1}{h^{k+1}} \sum_{i=1}^{k+1} (-1)^{i-1} \left[+ \int_{P^{top,i}_{\bar{x}}(h\bar{v}_1, \dots, h\bar{v}_{k+1})} \varphi - \int_{P^{bot,i}_{\bar{x}}(h\bar{v}_1, \dots, h\bar{v}_{k+1})} \varphi \right]
$$

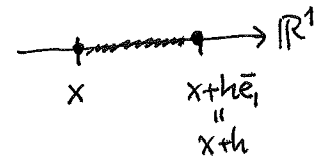
$k=0$

$k=0$
$n=1$

$$
d\varphi(P_x(\bar{e}_1)) = \lim_{h \to 0} \frac{1}{h} (-1)^0 \left[+ \int_{P^{top,1}_x(h\bar{e}_1)} \varphi - \int_{P^{bot,1}_x(h\bar{e}_1)} \varphi \right]
$$



$$
= \lim_{h \to 0} \frac{1}{h} \left(+\varphi(x+h) - \varphi(x) \right) = \varphi'(x)
$$



(113) It's much easier than it sounds to compute it!

THM 6.7.2: 1. If $\varphi = \sum_{1 \le i_1 < \dots < i_k \le n} a_{i_1 \dots i_k}(\bar{x})\, dx_{i_1} \wedge \dots \wedge dx_{i_k}$ with $a_{i_1 \dots i_k} \in C^2(U)$ then $d\varphi$ exists and $d\varphi \in A^{k+1}(U)$

4. A 0-form $\underset{f(\bar{x})}{\overset{\shortparallel}{f}} \in A^0(\underset{\mathbb{R}^n}{\overset{\cap}{U}})$ has $df = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(\bar{x})\, dx_i$

5. ~~[illegible]~~ More generally, one has $d(f\, dx_{i_1} \wedge \dots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\bar{x})\, dx_i \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

2. d is linear, i.e. $d(\varphi_1 + \varphi_2) = d\varphi_1 + d\varphi_2$

$$d(c\varphi) = c\, d\varphi \quad \forall c \in \mathbb{R}$$

EXAMPLES:

① Considering $\sin(xy) \in A^0(\mathbb{R}^2)$

$$d \sin(xy) = \frac{\partial \sin(xy)}{\partial x} \cdot dx + \frac{\partial \sin(xy)}{\partial y} \cdot dy$$

$$= y \cos(xy)\, dx + x \cos(xy) \cdot dy \in A^1(\mathbb{R}^2)$$

② ~~[illegible]~~ $\underbrace{x_1 x_3^2}_{\varphi :=} dx_1 \in A^1(\mathbb{R}^3)$

has $d\varphi = \left(\frac{\partial (x_1 x_3^2)}{\partial x_1} dx_1 + \frac{\partial (x_1 x_3^2)}{\partial x_2} dx_2 + \frac{\partial (x_1 x_3^2)}{\partial x_3} dx_3 \right) \wedge dx_1$

$$= x_3^2 \cancelto{0}{(dx_1 \wedge dx_1)} + 0 \cdot dx_2 \wedge dx_1 + 2x_1x_3\, dx_3 \wedge dx_1$$

$$= -2x_1x_3\, dx_1 \wedge dx_3 \in A^2(\mathbb{R}^3)$$

③ $\varphi = x_1 x_3^2\, dx_1 \wedge dx_2$

has $d\varphi = (x_3^2\, dx_1 + 0 \cdot dx_2 + 2x_1x_3\, dx_3) \wedge dx_1 \wedge dx_2$

$$= x_3^2 \underbrace{dx_1 \wedge dx_1}_{=0} \wedge dx_2 + 2x_1x_3\, dx_3 \wedge dx_1 \wedge dx_2$$

$$= + 2x_1x_3\, dx_1 \wedge dx_2 \wedge dx_3$$

(why plus?