Two more important properties of $\omega \overset{d}{\mapsto} d\omega$:

THM 6.7.7: 6.78: (i) For $\varphi \in A^k(U)$ having coefficients in $C^2(U)$, $d(d\varphi)=0$.

(ii) For $\varphi \in A^k(U)$ and $\psi \in A^l(U)$,

$$d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^k \varphi \wedge d\psi$$

proof: (ii) is EXER 6.7.11 on your HW!

For (i), it's enough to check it (by linearity of d) when

$$\varphi = f(\underline{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where one has

$$d(d\varphi) = d\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right)$$

$$= \underbrace{\left(\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i \right)}_{=0 \text{ since } dx_i \wedge dx_j = \begin{cases} 0 & \text{if } i=j \\ -dx_j \wedge dx_i & \text{if } i \neq j \end{cases} \text{ but } \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \quad \blacksquare$$

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$\boxed{\text{EXAMPLE: } d(d\, x^2y^3) = d(2xy^3 dx + 3x^2y^2 dy) = 6xy^2 dy\, dx + 6xy^2 dx\, dy = 0}$

§6.8 Div, grad, & curl

There are 3 operations in vector calculus of $\mathbb{R}^3$ and physics that have nice interpretations/unifications via $\omega \mapsto d\omega$:

For $f: \mathbb{R}^3 \to \mathbb{R}$, $\operatorname{grad} f := \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{bmatrix} = $ "∇f" $=$ gradient of f

For a vector field $\vec{F}: \mathbb{R}^3 \to \mathbb{R}$, $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$, $\operatorname{curl} F :=$ "$\nabla \times \vec{F}$" $= \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$ = curl of F

$$= \begin{bmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ -\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{bmatrix}$$

$\operatorname{div} F :=$ "$\nabla \bullet \vec{F}$" $= \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \bullet \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

= divergence of F

(117)

THM 6.8.3: (i) For $f \in A^0(\mathbb{R}^3)$, i.e. $f: \mathbb{R}^3 \to \mathbb{R}$,

then $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = W_{\nabla f}$ = the work 1-form of $\nabla f = \begin{bmatrix}\partial f/\partial x\\ \partial f/\partial y\\ \partial f/\partial z\end{bmatrix} \in A^1(\mathbb{R}^3)$

(ii) For $W_F = F_1\,dx + F_2\,dy + F_3\,dz \in A^1(\mathbb{R}^3)$
= Workform of $F = \begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix}$,

then $dW_F = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\wedge dy + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)dx\wedge dz + \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)dy\wedge dz$

$= \Phi_{\operatorname{curl} F}$ = the flux 2-form of curl $F \in A^2\mathbb{R}^3$

(iii) For $\Phi_F = F_1\,dy\wedge dz - F_2\,dx\wedge dz + F_3\,dx\wedge dy \in A^2(\mathbb{R}^3)$
= the flux 2-form of $F = \begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix}$,

then $d\Phi_F = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dx\wedge dy\wedge dz = M_{\operatorname{div} F}$
= the mass 3-form of div$F$ $\in A^3(\mathbb{R}^3)$

SUMMARY TABLE from p.630

$$\begin{array}{ccc}
\text{functions on } \mathbb{R}^3 & = & 0\text{-forms} \\
\downarrow \text{grad} & & \downarrow d \\
\text{vector fields in } \mathbb{R}^3 & \xrightarrow{\text{workform } W} & 1\text{-forms} \\
\downarrow \text{curl} & & \downarrow d \\
\text{vector fields in } \mathbb{R}^3 & \xrightarrow{\text{flux form } \Phi} & 2\text{-forms} \\
\downarrow \text{div} & & \downarrow d \\
\text{functions on } \mathbb{R}^3 & \xrightarrow{\text{mass form } M} & 3\text{-forms}
\end{array}$$

Note (RMK 6.8.6) that $d(d\varphi) = 0$ implies

$\operatorname{curl}(\operatorname{grad}(f)) = \begin{bmatrix}0\\0\\0\end{bmatrix}$ "gradient flows (or conservative vector fields) are irrotational"

$F = \nabla f$ for some f; curl $= 0$

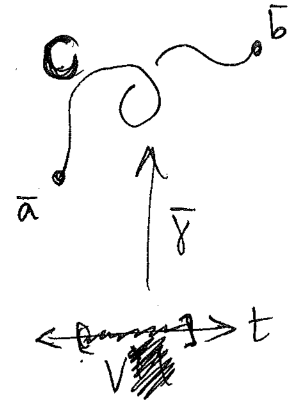
$\operatorname{div}(\operatorname{curl}(F)) = 0$ "a flow that comes from a curl is incompressible"

div $= 0$

(118) Why should a vector field $\bar{F} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$ of the form $\bar{F} = \nabla f$ for some function $f : \mathbb{R}^3 \to \mathbb{R}$ be called <u>conservative</u>?

Dipping into §6.10, the Fundamental Thm. for Line Integrals says

<u>THM 6.10.1</u>: For an oriented curve $C \subset \mathbb{R}^2$ or $\mathbb{R}^3$ and $f \in \mathring{A}(U)$ in some open neighborhood U of C will have



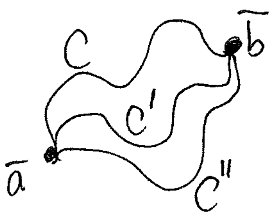
$$\int_C df = f(\bar{b}) - f(\bar{a}) \quad \left(= \int_{\partial C} f \right)$$

(i.e. Stokes's Thm!)

$$// \qquad \underbrace{\int \bar{F} \cdot \bar{\gamma}'(t)\, dt}_{\text{the line integral of } \bar{F} \text{ over } C}$$

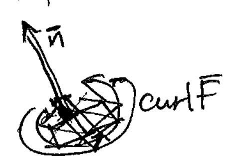
the answer only depends on the ~~starting~~ and endpoints $\bar{b}, \bar{a}$ and the difference in the value of the potential function $f(\bar{x})$ there, <u>not</u> the path $[C] = [\gamma(V)]$ taken to get there

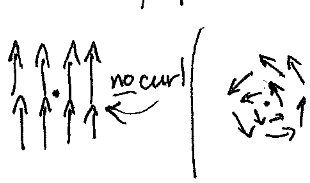
$\boxed{\text{conservative}}$



What does curl $\bar{F}$ "mean"?

The book discusses (pp. 631–632) the <u>curl probe</u>: put a paddle at $\bar{x} \in \mathbb{R}^3$ inside the flow with velocity vectors locally $\bar{F}(\bar{x}) = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$, and the <u>torque</u> locally exerted on the paddles will be proportional to $\operatorname{curl} \bar{F}(\bar{x}) \cdot \bar{n}$ if the handle points toward $\bar{n}$





<u>Why</u>? Call the paddle wheel blades $\bar{v}_1, \bar{v}_2$, so that $\bar{n} = \bar{v}_1 \times \bar{v}_2$

Then
$$\operatorname{curl} \bar{F}(\bar{x}) \cdot \bar{n} = \operatorname{curl} \bar{F}(\bar{x}) \cdot (\bar{v}_1 \times \bar{v}_2)$$

$$= \underbrace{\Phi_{\operatorname{curl} \bar{F}(\bar{x})}}_{= dW_{\bar{F}}} \left(P_{\bar{x}}(\bar{v}_1, \bar{v}_2) \right)$$

$$= \lim_{h \to 0} \frac{1}{h^2} \underbrace{\int_{\partial P_{\bar{x}}(h\bar{v}_1, h\bar{v}_2)} W_{\bar{F}}}_{\substack{\text{work done by } \bar{F} \text{ as a force field} \\ \text{going around the boundary of } P_{\bar{x}}(h\bar{v}_1, h\bar{v}_2)}}$$