(92) One can check that our ~~old~~ $dx_{i_1}\wedge\ldots\wedge dx_{i_k} = \big(\big((dx_{i_1}\wedge dx_{i_2})\wedge dx_{i_3}\big)\wedge\ldots\big)\wedge dx_{i_k}$ in this new sense.
(see EXAMPLES 6.1.14 for $k=3$)

In fact one doesn't need to worry about the parenthesization order:

PROP 6.1.15: $\varphi\wedge\omega$ has these properties:

Not obvious? (ignored in book) $\longrightarrow$ 0. $\varphi\wedge\omega\in A_c^{k+l}(\mathbb{R})$

EXER. 6.1.13 (let's skip it!)

(easy) 1. $\varphi\wedge(\omega_1+\omega_2) = \varphi\wedge\omega_1 + \varphi\wedge\omega_2$

(tricky) 2. $\varphi_1\wedge(\varphi_2\wedge\varphi_3) = (\varphi_1\wedge\varphi_2)\wedge\varphi_3$

(not hard) 3. $\varphi\wedge\omega = (-1)^{kl}\,\omega\wedge\varphi$ if $\varphi\in\Lambda_c^k(\mathbb{R}^n)$, $\omega\in\Lambda_c^l(\mathbb{R}^n)$

(e.g. $dx\wedge dy = -dy\wedge dx$
but $dx\wedge(dy\wedge dz) = +(dy\wedge dz)\wedge dx$)

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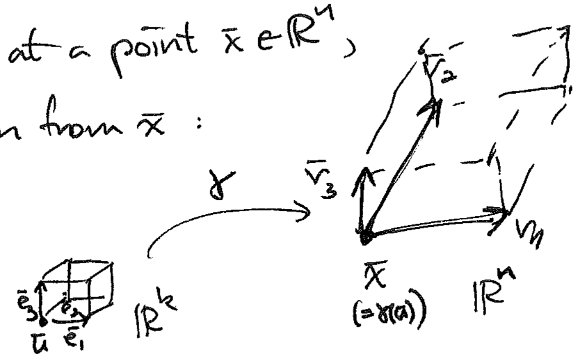
Most often, the $\bar v_1,\ldots,\bar v_k$ on which we evaluate a k-form are thought of as anchored at a point $\bar x\in\mathbb{R}^n$, spanning a parallelogram from $\bar x$:

$$P_{\bar x}(\bar v_1,\ldots,\bar v_k)$$



(think
$$\begin{array}{ccc} \mathbb{R}^k & & \mathbb{R}^n \\ \cup & & \cup \\ U & \xrightarrow{\gamma} & M \end{array}$$
$\bar u \longmapsto \bar x = \gamma(\bar u)$
$\bar e_1 \longmapsto D\gamma(\bar u)(\bar e_1) = \bar v_1$
$\vdots$
$\bar e_k \longmapsto D\gamma(\bar u)(\bar e_k)$)

So we'll want our k-form to have coefficients $a_{i_1\ldots i_k}(\bar x)$ that are functions of $\bar x$

DEF'N 6.1.16: A k-form field on $U\subset\mathbb{R}^n$ is a map $U\xrightarrow{\varphi} A_c^k(\mathbb{R}^n)$,

so $\displaystyle\varphi = \sum_{1\le i_1<\ldots<i_k\le n} a_{i_1\ldots i_k}(\bar x)\, dx_{i_1}\wedge\ldots\wedge dx_{i_k}$.

$A^k(U) := \{\text{all } \underbrace{k\text{-form fields}}_{} \text{ on } U\}$

sometimes called "differential k-forms" (on U)

EXAMPLE (6.1.7) $\varphi = \cos(xz)\,dx\wedge dy \in A^2(\mathbb{R}^3)$

with $\varphi\left(P_{\begin{pmatrix}1\\2\\\pi\end{pmatrix}}\left(\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}2\\2\\3\end{bmatrix}\right)\right) = \cos(1\cdot\pi)\det\begin{bmatrix}1&2\\0&2\end{bmatrix} = -2$

$\varphi\left(P_{\begin{pmatrix}1/2\\2\\\pi\end{pmatrix}}\left(\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}2\\2\\3\end{bmatrix}\right)\right) = \cos\left(\tfrac{1}{2}\cdot\pi\right)\det\begin{bmatrix}1&2\\0&2\end{bmatrix} = 0$

§6.2 Integrating form-fields over (parametrized) manifolds

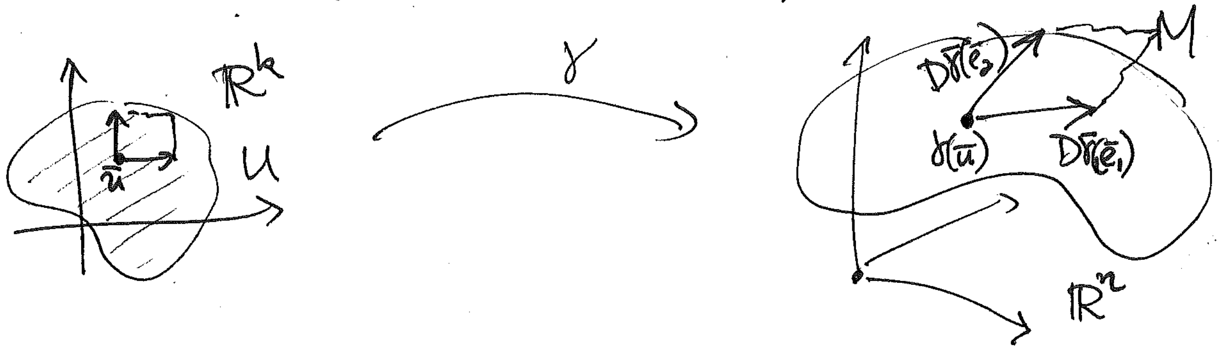
Given a (nice) (§5.2) parametrization $\begin{array}{ccc} U & \xrightarrow{\gamma} & M \\ \cap & & \cap \\ \mathbb{R}^k & & \mathbb{R}^n \end{array}$ for a k-dim'l manifold M,

and a k-form field $\varphi \in A^k(\mathbb{R}^n)$, we now have the correct set-up for integration that pays attention to <u>the orientation M gets from U, γ</u>:

DEF'N 6.2.1: In this setting, the integral of the k-form field φ over $M = \overrightarrow{[\gamma(U)]}$

is $\displaystyle\int_M \varphi := \int_U \varphi\Big(P_{\gamma(\bar{u})}\big(\underbrace{D\gamma(\bar{u})}_{(D\gamma_1(\bar{u}),\ldots,D\gamma_n(\bar{u}))}\big)\Big)\, |d^k \bar{u}|$

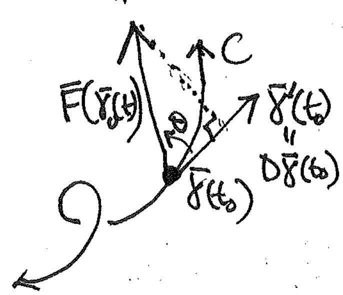
the orientation on M coming from $U \xrightarrow{\gamma} M$



EXAMPLE: $k=1$ (anticipating §6.5)

A 1-form field $\varphi = \sum_{i=1}^{n} F_i(\bar{x})\, dx_i = F_1(\bar{x})\,dx_1 + F_2(\bar{x})\,dx_2 + \ldots + F_n(\bar{x})\,dx_n$
$\in A^1(\mathbb{R}^n)$

is the same as a <u>vector field</u> $\bar{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$
(think of it as a <u>force field</u>) $\bar{x} \longmapsto \bar{F}(\bar{x}) = \begin{bmatrix} F_1(\bar{x}) \\ F_2(\bar{x}) \\ \vdots \\ F_n(\bar{x}) \end{bmatrix}$



and if our 1-manifold (curve) C is parametrized $\begin{array}{ccc} U & \xrightarrow{\bar{\gamma}} & \mathbb{R}^n \\ \cap & & \\ \mathbb{R}^1 & & \\ t & \longmapsto & \bar{\gamma}(t) \end{array}$

then at some point $\gamma(t_0)$ on C, the component of the force $\bar{F}(\bar{\gamma}_0(t))$ in the direction of the tangent vector $\bar{\gamma}'(t_0)$ is $|F(\bar{\gamma}(t_0))| \cos\theta$, so one can re-interpret

$$\int_C \varphi = \int_U \varphi\left(P_{\bar\gamma(t)}(D\bar\gamma(t))\right) |d^1 t| \qquad D\bar\gamma(t) = \sum_{i=1}^n \gamma_i'(t)\bar e_i = \begin{bmatrix}\gamma_1'(t)\\ \gamma_2'(t)\\ \vdots \\ \gamma_n'(t)\end{bmatrix}$$

$$= \int_U \left(\sum_{i=1}^n F_i(\bar\gamma(t))\,dx_i\right)\left(\gamma_1'(t),\dots,\gamma_n'(t)\right) |d^1 t|$$

$$= \int_U \begin{bmatrix}F_1(\bar\gamma(t))\\ \vdots \\ F_n(\bar\gamma(t))\end{bmatrix} \bullet \begin{bmatrix}\gamma_1'(t)\\ \vdots \\ \gamma_n'(t)\end{bmatrix} |d^1 t|$$

$$= \int_U \bar F(\bar\gamma(t)) \bullet \bar\gamma'(t)\ |d^1 t| \quad \overset{\text{DEF'N}}{=:} \text{ line integral of the vector field } \bar F \text{ over the curve } C = \bar\gamma(U)$$

called the line integral of $\bar F$ over C

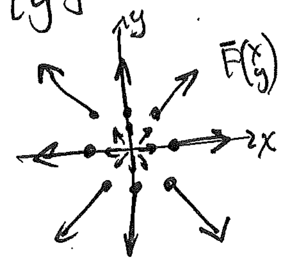
$$= \int_U \underbrace{|\bar F(\bar\gamma(t))| \cos\theta}_{\substack{\text{component of } \bar F(\bar\gamma(t)) \\ \text{in direction } \bar\gamma'(t)}} \cdot \underbrace{|\bar\gamma'(t)|\,|d^1 t|}_{\substack{\text{arclength} \\ \text{integration factor} \\ \text{for } C}} = \text{work done by } \bar F \text{ on a particle traveling along } C \text{ via the parametrization } \bar\gamma.$$

$\boxed{\text{DEF'N 6.5.1: } \varphi = F_1(\bar x)dx_1 + \dots + F_n(\bar x)dx_n \text{ is called the work form } W_{\bar F} \text{ of } \bar F}$

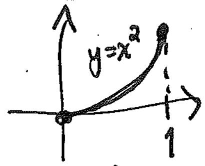
---

e.g. Let's compute for the radial vector field $\bar F\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$ on $\mathbb{R}^2$

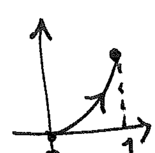
$\varphi = x\,dx + y\,dy \in A^2(\mathbb{R}^2)$

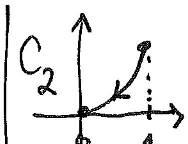


the work done on a particle traveling along the curve

C = 

parametrized forward $U_1 \xrightarrow{\gamma_1} \mathbb{R}^2$, $U_1 = [0,1]$, $t \longmapsto \begin{pmatrix}t\\t^2\end{pmatrix}$ and backward $U_2 \xrightarrow{\gamma_2} \mathbb{R}^2$, $U_2 = [0,1]$, $t \longmapsto \begin{pmatrix}1-t\\(1-t)^2\end{pmatrix}$

C_1 

C_2 

$$\int_{C_1}\varphi = \int_U \varphi(P_{\bar\gamma(t)}(D\bar\gamma(t)))|d^1t| = \int_{t=0}^{t=1}(t\,dx + t^2 dy)\begin{bmatrix}1\\2t\end{bmatrix}|dt| = \int_0^1 (t\cdot 1 + t^2\cdot 2t)\,dt$$

$$= \int_0^1 (t + 2t^3)\,dt = \left[\frac{t^2}{2} + \frac{t^4}{2}\right]_0^1 = 1$$

$$\int_{C_2}\varphi = \int_0^1 ((1-t)dx + (1-t)^2 dy)\begin{bmatrix}-1\\ \dots\end{bmatrix}|d^1t| = \int_0^1 (-(1-t) - 2(1-t)^3)\,dt$$

$$= -1 = -\int_{C_1}\varphi$$